

# EFFECTIVE CIRCLE COUNT FOR APOLLONIAN PACKINGS AND CLOSED HOROSPHERES

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**ABSTRACT.** The main result of this paper is an effective count for Apollonian circle packings that are either bounded or contain two parallel lines. We obtain this by proving an effective equidistribution of closed horospheres in the unit tangent bundle of a geometrically finite hyperbolic 3-manifold, whose fundamental group has critical exponent bigger than 1. We also discuss applications to Affine sieves. Analogous results for surfaces are treated as well.

## 1. INTRODUCTION

**1.1. Apollonian circle packings.** An Apollonian circle packing is an ancient Greek construction which is made by repeatedly inscribing circles into the triangular interstices of four mutually tangent circles in the plane. In recent years, there have been many new and exciting developments in the study of Apollonian circle packings which we refer to [15], [31] and [24] for references.

The main goal of this paper is to obtain an effective version of the counting theorem for circles in an Apollonian packing with bounded curvature.

Let  $\mathcal{P}$  be an Apollonian circle packing, that is either bounded or lies between two parallel lines (i.e., congruent to the packing in Figure 2). For  $T > 0$  and  $\mathcal{P}$  bounded, we define the following circle counting function

$$N_T(\mathcal{P}) := \#\{C \in \mathcal{P} : \text{Curv}(C) < T\}$$

where  $\text{Curv}(C)$  denotes the curvature of  $C$ , i.e., the reciprocal of the radius of  $C$ . For  $\mathcal{P}$  unbounded between two parallel lines, we adjust the definition of  $N_T(\mathcal{P})$  to count circles only in a fixed period.

The main term in the asymptotic for  $N_T(\mathcal{P})$  will be described in terms of the residual set of  $\mathcal{P}$  (=the closure of the union of all circles in  $\mathcal{P}$ ), denoted by  $\text{Res}(\mathcal{P})$ . We denote by  $\alpha$  the Hausdorff dimension of  $\text{Res}(\mathcal{P})$ ;  $\alpha$  is independent of  $\mathcal{P}$  and known to be approximately

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1.30568(8) [23]. Let  $\mathcal{H}^\alpha(\text{Res}(\mathcal{P}))$  be the  $\alpha$ -dimensional Hausdorff measure of  $\text{Res}(\mathcal{P})$  for bounded  $\mathcal{P}$ . For  $\mathcal{P}$  between two parallel lines, we let  $\mathcal{H}^\alpha(\text{Res}(\mathcal{P}))$  be the measure of  $\text{Res}(\mathcal{P})$  in a fixed period.

The error term in our asymptotic formula depends directly on the  $L^2$ -spectral gap of the complete hyperbolic 3 manifold whose fundamental group is the symmetry group of  $\mathcal{P}$ . The group  $\text{PSL}_2(\mathbb{C})$  acts on the extended plane by linear fractional transformations. Set

$$\mathcal{A}_{\mathcal{P}} := \{g \in \text{PSL}_2(\mathbb{C}) : g(\mathcal{P}) = \mathcal{P}\}.$$

It is known that  $\mathcal{A}_{\mathcal{P}}$  is a geometrically finite discrete subgroup of  $\text{PSL}_2(\mathbb{C})$  with critical exponent equal to  $\alpha$  (cf. [19]). The fact  $\alpha > 1$  yields that  $\alpha(2-\alpha)$  is the smallest eigenvalue of the Laplacian  $\Delta$  on the  $L^2$ -spectrum of the hyperbolic manifold  $\mathcal{A}_{\mathcal{P}} \backslash \mathbb{H}^3$  by Sullivan [35] and is also isolated by Lax and Phillips [21]. Hence there exists  $1 < s_1 < \alpha$  such that there is no eigenvalue of  $\Delta$  in  $L^2(\mathcal{A}_{\mathcal{P}} \backslash \mathbb{H}^3)$  between  $\alpha(2-\alpha)$  and  $s_1(2-s_1)$ . Since all  $\mathcal{A}_{\mathcal{P}}$ 's are conjugate to each other by elements of  $\text{PSL}_2(\mathbb{C})$ ,  $s_1$  is independent of  $\mathcal{P}$ .

Our effective counting result can be stated as follows:

**Theorem 1.1.** *As  $T \rightarrow \infty$ ,*

$$N_T(\mathcal{P}) = c_A \cdot \mathcal{H}^\alpha(\text{Res}(\mathcal{P})) \cdot T^\alpha + O(T^{\alpha - \frac{2(\alpha-s_1)}{63}})$$

*where  $c_A > 0$  is a constant independent of  $\mathcal{P}$ .*

**Remark 1.2.** (1) In [19], the asymptotic  $N_T(\mathcal{P}) \sim c_{\mathcal{P}} \cdot T^\alpha$  was obtained with less clear interpretation of the main term.

(2) A similar type of asymptotic formula was obtained in [26] for all Apollonian packings (whether bounded or not) by counting circles in a bounded region, but with no error term.

(3) There are several different ways of understanding the main term  $c_A \cdot \mathcal{H}^\alpha(\text{Res}(\mathcal{P}))$  stemming from different approaches to the counting problem. One description is given in our paper (see (8.8)). The aforementioned paper [26] gives another expression for the main term as well.

(4) An Apollonian packing  $\mathcal{P}$  is called integral if the curvatures of all circles in  $\mathcal{P}$  are integers. Any integral Apollonian packing is known to be either bounded or lies between two parallel lines. Therefore Theorem 1.1 applies to all integral Apollonian packings.

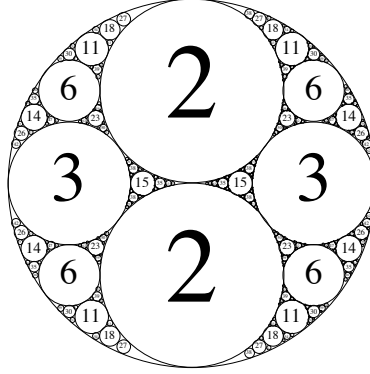


FIGURE 1. A bounded Apollonian circle packing.

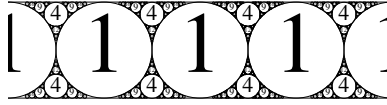


FIGURE 2. An unbounded Apollonian circle packing bounded by two parallel line.

Based on the Descartes circle theorem [7], the approach in [19] was to relate the circle counting problem with the equidistribution of expanding closed horospheres in the unit tangent bundle of the hyperbolic 3-manifold  $\mathcal{A}_P \backslash \mathbb{H}^3$ .

The new achievement of this paper is an *effective* equidistribution of closed horospheres (Theorem 1.3). Besides its application to counting problems, such equidistribution result is of independent interest in homogeneous dynamics.

**1.2. Effective equidistribution of closed horospheres.** We obtain an effective equidistribution for closed horospheres in the unit tangent bundle of hyperbolic  $n$ -manifolds for  $n = 2$  or  $3$ . Consider the upper half space  $\mathbb{H}^n = \{(x, y) : x \in \mathbb{R}^{n-1}, y > 0\}$  and let  $G$  denote the group of orientation preserving isometries of  $\mathbb{H}^n$ . That is,  $G = \text{PSL}_2(\mathbb{R})$  for  $n = 2$  and  $G = \text{PSL}_2(\mathbb{C})$  for  $n = 3$ .

Let  $\Gamma < G$  be a torsion-free discrete subgroup, which is not virtually abelian. We assume that  $\Gamma$  is geometrically finite, that is, it admits a finite sided fundamental domain in  $\mathbb{H}^n$ . The limit set  $\Lambda(\Gamma)$  is the subset of the boundary  $\partial(\mathbb{H}^n) = \mathbb{R}^n \cup \{\infty\}$  consisting of all accumulation points in an orbit  $\Gamma(z)$ ,  $z \in \mathbb{H}^n$ . We denote by  $0 < \delta \leq n - 1$  the

critical exponent of  $\Gamma$ ; it is equal to the Hausdorff dimension of  $\Lambda(\Gamma)$  [36].

For  $G = \mathrm{PSL}_2(\mathbb{R})$ , set  $K := \mathrm{PSO}(2)$ , and for  $G = \mathrm{PSL}_2(\mathbb{C})$ , set  $K := \mathrm{PSU}(2)$ . In both cases, set

$$A := \left\{ a_y := \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} : y > 0 \right\},$$

and let  $M$  be the centralizer of  $A$  in  $K$ .

The hyperbolic manifold  $\Gamma \backslash \mathbb{H}^n$  and its unit tangent bundle  $T^1(\Gamma \backslash \mathbb{H}^n)$  can be identified with the double quotient spaces  $\Gamma \backslash G/K$  and  $\Gamma \backslash G/M$  respectively. Accordingly, functions on  $\Gamma \backslash \mathbb{H}^n$  (resp.  $T^1(\Gamma \backslash \mathbb{H}^n)$ ) can be considered as right  $K$ -invariant (resp.  $M$ -invariant) functions on  $\Gamma \backslash G$ . Since  $a_y$  commutes with  $M$ ,  $a_y$  acts on  $\Gamma \backslash G/M$  by the multiplication from the right and this action corresponds to the geodesic flow in the unit tangent bundle.

Set  $N = \{g \in G : a_y^{-1}ga_y \rightarrow e \text{ as } y \rightarrow \infty\}$ ; the contracting horospherical subgroup under the action of  $a_y$ . Setting

$$n_x := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

we have  $N = \{n_x : x \in \mathbb{R}\}$  for  $G = \mathrm{PSL}_2(\mathbb{R})$ , and  $N = \{n_x : x \in \mathbb{C}\}$  for  $G = \mathrm{PSL}_2(\mathbb{C})$ . Even though there is no action of  $N$  on  $\Gamma \backslash G/M$ , the  $N$ -orbits  $\{[g]N := \Gamma \backslash \Gamma gMN/M : g \in G\}$  are well-defined since  $N$  is normalized by  $M$ ; these orbits give rise to the stable horospherical foliation of  $T^1(\Gamma \backslash \mathbb{H}^n)$ .

In the rest of the introduction, we assume that  $(n-1)/2 < \delta < n-1$  and that  $\Gamma \backslash \Gamma N$  is closed in  $\Gamma \backslash G$ . In particular,  $\Gamma$  is of infinite covolume in  $G$ . By the torsion-free assumption on  $\Gamma$ ,  $\Gamma \cap NM = \Gamma \cap N$  and we can identify  $\Gamma \backslash \Gamma NM/M$  with  $(\Gamma \cap N) \backslash N$ . Note that the quotient  $(\Gamma \cap N) \backslash N$  can be naturally identified with  $(\mathbb{R}/\mathbb{Z})^k \times \mathbb{R}^{n-1-k}$  where  $k$  denotes the rank of the free abelian subgroup  $\Gamma \cap N$ .

**1.3. Equidistribution in spectral terms.** We describe the effective equidistribution of  $\Gamma \backslash \Gamma Na_y$  as  $y \rightarrow 0$  in  $T^1(\Gamma \backslash \mathbb{H}^n)$  in terms of the  $M$ -invariant spectrum of  $L^2(\Gamma \backslash G)$  for a Casimir element of  $G$ .

By Lax and Phillips [21] and Sullivan [35], the Laplacian  $\Delta$  on  $L^2(\Gamma \backslash \mathbb{H}^n)$  has only finitely many eigenvalues

$$0 < \alpha_0 = \delta(n-1-\delta) < \alpha_1 \leq \cdots \leq \alpha_k < 1$$

lying below the continuous spectrum  $[\frac{(n-1)^2}{4}, \infty)$ . The existence of a point eigenvalue is the precise reason that our main theorem requires

the condition  $\delta > (n-1)/2$ . Writing  $\alpha_1 = s_1(n-1-s_1)$ , any positive number

$$0 < \mathbf{s}_\Gamma < \delta - s_1$$

will be referred to as a *spectral gap* of  $\Gamma$ .

Let  $\mathcal{C}$  denote the element in the center of the universal enveloping algebra of the Lie algebra of  $G$ , which acts on  $K$ -invariant smooth functions as the negative Laplacian  $-\Delta$ . So  $\mathcal{C}$  is a Casimir element of  $G$ , up to a scalar multiple. Then  $L^2(\Gamma \backslash G)$  contains the unique irreducible infinite dimensional subrepresentation  $V$  (a complementary series representation) on which  $\mathcal{C}$  acts by the scalar  $\delta(\delta - n + 1)$ .

Let  $\mathcal{C}_K$  denote a Casimir element of  $K$  and  $\hat{K}$  the unitary dual of  $K$ , that is, the equivalence classes of all irreducible unitary representations of  $K$ . For  $n = 2$ ,  $\hat{K}$  can be parametrized by  $\mathbb{Z}$  so that  $\ell \in \hat{K}$  corresponds to the one dimensional representation  $V_\ell$  on which  $\mathcal{C}_K$  acts by the scalar  $-4\ell^2$ . For  $n = 3$ ,  $\hat{K}$  can be parametrized by  $\mathbb{Z}_{\geq 0}$  so that  $\ell \in \hat{K}$  corresponds to the  $2\ell + 1$  dimensional representation  $V_\ell$  where  $\mathcal{C}_K$  acts by the scalar  $-\ell(\ell + 1)$ .

The representation  $V$  is decomposed into the orthogonal sum  $\oplus_{\ell \in \hat{K}} V_\ell$  with the subspace  $V_\ell^M$  of  $M$ -invariant vectors being one dimensional. Let  $\phi_\ell \in C^\infty(\Gamma \backslash G) \cap L^2(\Gamma \backslash G)$  be a unit vector in  $V_\ell^M$  for each  $\ell \in \hat{K}$ . We show that there exists  $c_n(\ell) \neq 0$  such that for all  $y > 0$ ,

$$\int_{n_x \in (N \cap \Gamma) \backslash N} \phi_\ell(n_x a_y) dx = c_n(\ell) \cdot y^{n-1-\delta}.$$

The inner product  $\langle \psi_1, \psi_2 \rangle$  in  $L^2(\Gamma \backslash G)$  is given by

$$\langle \psi_1, \psi_2 \rangle = \int_{\Gamma \backslash G} \psi_1(g) \overline{\psi_2(g)} dg$$

where  $dg$  denotes a  $G$ -invariant measure on  $\Gamma \backslash G$ .

The following is our main theorem on the effective equidistribution:

**Theorem 1.3.** *Let  $n = 2$  or  $3$ . Let  $(n-1)/2 < \delta < n-1$ . For any  $\psi \in C_c^\infty(\Gamma \backslash G)^M$ , as  $y \rightarrow 0$ ,*

$$\begin{aligned} & \int_{(N \cap \Gamma) \backslash N} \psi(n_x a_y) dx \\ &= \sum_{\ell \in \hat{K}} c_n(\ell) \cdot \langle \psi, \phi_\ell \rangle \cdot y^{n-1-\delta} + O(\mathcal{S}_{2n-1}(\psi) \cdot y^{(n-1-\delta) + \frac{2\mathbf{s}_\Gamma}{2n+1}}) \end{aligned}$$

where  $\mathcal{S}_{2n-1}(\psi)$  denotes the  $L^2$ -Sobolev norm of  $\psi$  of order  $2n - 1$ . Moreover

$$c_n(\ell) = O((|\ell| + 1)^{(n-2)/2}) \quad \text{and} \quad \sum_{\ell \in \hat{K}} |c_n(\ell) \langle \psi, \phi_\ell \rangle| = O(\mathcal{S}_2(\psi)).$$

- Remark 1.4.** (1) When  $\Gamma$  is a lattice in  $G$ , i.e., when  $\delta = n - 1$ , an effective equidistribution for expanding closed horospheres is well known, via the mixing of the geodesic flow and the thickening argument. This argument goes back to the 1970 thesis of Margulis [22] and was generalized by Eskin and McMullen [12]. We also refer to [33] for a sharp effective result in this setting for the dimension 2 case.
- (2) In principle, our methods should extend to prove an analogous result for  $G = \text{Isom}^+(\mathbb{H}^n)$  for any  $n \geq 2$ ; however computations needed in the proofs of Theorems 2.6 and 3.5 seem very intricate and it is not clear what their higher dimensional generalizations are.
- (3) When  $\psi$  is  $K$ -invariant, Theorem 1.3 was obtained in [19]. See also [17] for its extensions to other rank one Lie groups.

A non-effective version of Theorem 1.3 was obtained in [29] when  $(N \cap \Gamma) \backslash N$  is compact and in [25] in general, with no restriction on  $\delta > 0$ . In these papers, the main term was given in terms of the Burger-Roblin measure associated to the stable horospherical foliation. In applications, it is much handier to have the main term in terms of a measure instead of the above type of infinite sum. For this reason, we present an alternative formulation of Theorem 1.3 in the following.

**1.4. Equidistribution in ergodic terms.** Let  $\nu_j$  denote the Patterson-Sullivan measure on the limit set  $\Lambda(\Gamma)$  associated to the basepoint  $j = (0_{n-1}, 1) \in \mathbb{H}^n$ , which is unique up to a constant multiple.

Sullivan gave an explicit construction of the base eigenfunction  $\phi_0 \in L^2(\Gamma \backslash G)^K$  using  $\nu_j$ :

$$\phi_0(n_x a_y) = \int_{u \in \mathbb{R}^{n-1}} \left( \frac{(|u|^2 + 1)y}{|x - u|^2 + y^2} \right)^\delta d\nu_j(u). \quad (1.5)$$

Here and also later, we identify  $\mathbb{C} = \mathbb{R}^2$  for  $n = 3$ , so that  $|x - u|^2 = (x_1 - u_1)^2 + (x_2 - u_2)^2$  for  $x = x_1 + ix_2$  and  $u = (u_1, u_2)$ . We normalize  $\nu_j$  so that  $\|\phi_0\|_2 = 1$  [36].

Define the measure  $\tilde{m}_N^{\text{BR}}$  on  $G$  in the Iwasawa coordinates  $G = KAN$ : for  $\psi \in C_c(G)$ ,

$$\tilde{m}_N^{\text{BR}}(\psi) = \int_{KAN} \psi(ka_y n_x) y^{\delta-1} dx dy d\nu_j(k(0)).$$

This measure is left  $\Gamma$ -invariant and right  $N$ -invariant, and the BR measure  $m_N^{\text{BR}}$  (associated to the stable horospherical subgroup  $N$ ) is the measure on  $\Gamma \backslash G$  induced from  $\tilde{m}_N^{\text{BR}}$ . The BR measure is an infinite measure whenever  $0 < \delta < n - 1$  [25].

**Theorem 1.6.** *Let  $(n - 1)/2 < \delta \leq n - 1$ . For any  $\psi \in C_c^\infty(\Gamma \backslash G)^M$ , as  $y \rightarrow 0$ ,*

$$\begin{aligned} \int_{(N \cap \Gamma) \backslash N} \psi(n_x a_y) dx &= \kappa_\Gamma \cdot m_N^{\text{BR}}(\psi) \cdot y^{n-1-\delta} \\ &\quad + O(\mathcal{S}_{2n-1}(\psi) \cdot y^{(n-1-\delta) + \frac{2\mathbf{s}_\Gamma}{2n+1}}) \end{aligned}$$

where  $\kappa_\Gamma = \int_{x \in \mathbb{R}^{n-1}} (1 + |x|^2)^{-\delta} dx \cdot \int_{n_x \in (N \cap \Gamma) \backslash N} (1 + |x|^2)^\delta d\nu_j(x)$ .

**1.5. Effective orbital counting and Affine sieves in sectors.** Let  $Q$  be a quadratic form over  $\mathbb{Q}$  of signature  $(n, 1)$  and  $v_0 \in \mathbb{Z}^{n+1}$  a non-zero vector such that  $Q(v_0) = 0$ . Let  $G_0$  denote the identity component of  $\text{SO}_Q(\mathbb{R})$ . As well known,  $G_0$  is isomorphic to  $\text{PSL}_2(\mathbb{R})$  (for  $n = 2$ ) and  $\text{PSL}_2(\mathbb{C})$  (for  $n = 3$ ) as real Lie groups. Let  $\Gamma < G_0(\mathbb{Z})$  be a geometrically finite subgroup with  $\delta > (n - 1)/2$  such that  $v_0\Gamma$  is discrete. For each square-free integer  $d$ , let  $\Gamma_d < \Gamma$  be a subgroup containing  $\{\gamma \in \Gamma : \gamma \equiv I \pmod{d}\}$  and  $\text{Stab}_\Gamma v_0 = \text{Stab}_{\Gamma_d} v_0$ .

In order to describe counting theorems of  $v_0\Gamma$  in sectors of the cone  $\{Q = 0\}$ , we consider the representation  $G \rightarrow G_0$  such that  $N$  is contained in  $\text{Stab}_G(v_0)$ .

Fix a norm  $\|\cdot\|$  on  $\mathbb{R}^{n+1}$ . For any subset  $\Omega \subset K$  and  $T > 0$ , define the sector

$$S_T(\Omega) := \{v \in v_0 A \Omega : \|v\| < T\}.$$

By a theorem of Bourgain, Gamburd and Sarnak [3], there exists a uniform spectral gap, say  $\mathbf{s}_0 > 0$ , for all  $\Gamma_d$ ,  $d$  square-free. We deduce from Theorem 1.3:

**Theorem 1.7.** *Suppose that the boundary of  $\Omega^{-1}(0)$  is a proper algebraic subset of  $\mathbb{R}^n \cup \{\infty\}$ . Then for any  $\gamma \in \Gamma$ , as  $T \rightarrow \infty$ ,*

$$\#\{v \in v_0 \Gamma_d \gamma \cap S_T(\Omega)\} = \frac{\Xi_{v_0}(\Gamma, \Omega)}{[\Gamma : \Gamma_d]} \cdot T^\delta + O(T^{\delta - \frac{8\mathbf{s}_0}{n(n+9)(2n+1)}}).$$

Identifying  $\Gamma$  with its pull back in  $G$ ,  $\Xi_{v_0}(\Gamma, \Omega)$  is given by

$$\Xi_{v_0}(\Gamma, \Omega) = \kappa_\Gamma \int_{k \in \Omega^{-1}} \frac{d\nu_j(k(0))}{\|v_0 k^{-1}\|^\delta}. \quad (1.8)$$

As  $\nu_j$  is supported on the limit set  $\Lambda(\Gamma)$ ,  $\Xi_{v_0}(\Gamma, \Omega) > 0$  if and only if the interior of  $\Omega^{-1}(0)$  intersects  $\Lambda(\Gamma)$ .

Theorem 1.7 has an application in studying almost prime vectors in the orbit of  $\Gamma$ , lying in a fixed sector. For instance, the following theorem can be deduced from Theorem 1.7 using the same analysis as in [19, section 8].

**Theorem 1.9.** *Suppose that  $\Omega^{-1}(0)$  intersects  $\Lambda(\Gamma)$ . Then there exists  $R \geq 1$  (depending on  $\mathbf{s}_0$ ) such that for each  $1 \leq i \leq (n+1)$ ,*

$$\#\{\mathbf{x} \in v_0\Gamma \cap S_T(\Omega) : x_1 \cdots x_i \text{ has at most } R \text{ prime factors}\} \asymp \frac{T^\delta}{(\log T)^i}$$

where  $f(T) \asymp g(T)$  means that their ratio is between two positive constants uniformly for all  $T \gg 1$ .

Theorem 1.7 is proved in [25] without an error term. When the norm is  $K$ -invariant and  $\Omega = K$ , Theorem 1.7 was proved in [19]. Theorem 1.9 for  $\Omega = K$  has been obtained in [19] (also see [20]).

**1.6. Organization:** In section 2, we find a *computable* recursive formula (Theorem 2.6) for a raising operator among  $M$ -invariant vectors in a general complementary series representation of  $G$ . Using this, in section 3, we obtain an explicit description of  $\phi_\ell$ 's which turn out to be related to the Legendre polynomials for  $n = 3$ . Understanding each  $\phi_\ell$  as a function of  $\Gamma \backslash G$ , rather than as a vector in the Hilbert space  $L^2(\Gamma \backslash G)$ , is crucial in our approach, as we need to deal with several convergence issues of the integrals of  $\phi_\ell$ 's as well as thickening the  $N$ -integrals of  $\phi_\ell$ 's uniformly over all  $\ell$ 's. In section 4, we compute the  $N$ -integrals of  $\phi_\ell$ 's and compute  $c_\ell$ 's explicitly (modulo  $c_0$ ). In section 5, we carry out the thickening of the  $N$ -integrals of  $\phi_\ell$ 's uniformly. Since  $\phi_\ell$ 's are not supported on compact subsets of  $\Gamma \backslash G$ , this step is delicate, as we need to ensure that there is at most a polynomial error term in  $\ell$  in this procedure. Theorems 1.3 and 1.6 are proved in section 6 and 7 respectively. In section 8, we deduce Theorem 1.7 and Theorem 1.1 from Theorem 1.6.

Added in print: Soon after we submitted the first version of our paper to the arXive, we received a preprint by Vinogradov [37], which also proves Theorem 1.1 (with a weaker error term) using different methods. His main term is same as (8.8).



**Acknowledgment:** We thank Peter Sarnak for useful comments on the preliminary version of this paper.

## 2. LADDER OPERATORS

Let  $G$  be  $\mathrm{PSL}_2(\mathbb{R})$  or  $\mathrm{PSL}_2(\mathbb{C})$ . Hence  $G$  is isomorphic to the identity component of  $\mathrm{SO}(n, 1)$  for  $n = 2$  and  $3$  respectively.

Let  $K$  be a maximal compact subgroup of  $G$ ,  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$  the corresponding Cartan decomposition of the Lie algebra  $\mathfrak{g}$ , and let  $B$  be the Killing form for  $\mathfrak{g}$ . Let  $A = \exp(\mathfrak{a})$  where  $\mathfrak{a}$  is a maximal abelian subspace of  $\mathfrak{p}$  (of dimension 1) and  $M$  the centralizer of  $A$  in  $K$ . The notation  $V^K$  means the subspace of  $K$ -invariant vectors and  $V^M$  is defined similarly.

Let  $V$  be an infinite dimensional irreducible unitary representation of  $G$  with a non-zero  $K$ -fixed vector. We fix  $v_0$  a unit vector in  $V^K$ , which is unique up to a scalar multiple.

Let  $V^\infty$  denote the set of smooth vectors of  $V$ , i.e.,  $v \in V^\infty$  if the map  $g \mapsto gv$  is a smooth function  $G \rightarrow V$ . Every element of  $\mathfrak{g}$  acts as a differential operator on  $V^\infty$ : for  $X \in \mathfrak{g}$  and  $v \in V^\infty$ ,

$$\pi(X)(v) := \left. \frac{d}{dt}(\exp(tX).v) \right|_{t=0}$$

where  $\exp(X) = \sum_{j=0}^{\infty} \frac{X^j}{j!}$  denotes the usual exponential map  $\mathfrak{g} \rightarrow G$ . This action extends to the action of the universal enveloping algebra  $U(\mathfrak{g})$  on  $V^\infty$ .

Let  $\{X_i\}$  be a basis for  $\mathfrak{k}$  and  $\{Y_i\}$  be a basis for  $\mathfrak{p}$  such that  $B(X_i, X_j) = -\delta_{ij}$  and  $B(Y_i, Y_j) = \delta_{ij}$ . We set

$$\mathcal{C} = \frac{1}{n-1}(-\sum X_i^2 + \sum Y_i^2) \quad \text{and} \quad \mathcal{C}_K = \frac{1}{(n-1)^2}(-\sum X_i^2);$$

hence they are the Casimir operators of  $G$  and  $K$  respectively, up to constant multiples.

As  $V$  is irreducible,  $\mathcal{C}$  acts on  $V$  as a scalar, say,  $\lambda$ . We will assume that

$$\lambda = s(s - n + 1)$$

where  $\frac{n-1}{2} < s < (n-1)$ . That is,  $V$  is a complementary series representation of  $G$ .

If  $n = 2$ , the unitary dual  $\hat{K}$  can be parametrized by  $\mathbb{Z}$  so that  $\ell \in \hat{K}$  corresponds to the one dimensional representation where  $\mathcal{C}_K$  acts by the scalar  $-4\ell^2$ . For  $n = 3$ , the unitary dual  $\hat{K}$  can be parametrized by  $\mathbb{Z}_{\geq 0}$  so that  $\ell \in \hat{K}$  corresponds to the  $2\ell + 1$  dimensional representation where  $\mathcal{C}_K$  acts by the scalar  $-\ell(\ell + 1)$ .

As a  $K$ -representation, we write

$$V = \oplus_{\ell \in \hat{K}} m_\ell V_\ell$$

where the multiplicity  $m_\ell$  of  $V_\ell$  is at most one for each  $\ell$  (see the remark following Theorem 4.5 of [38]). We also have that the space  $V_\ell^M$  is at most one dimensional [10].

For  $G = \mathrm{PSL}_2(\mathbb{R})$ , set  $K = \mathrm{PSO}(2)$ , and for  $G = \mathrm{PSL}_2(\mathbb{C})$ , we set  $K = \mathrm{PSU}(2)$ . Let  $A$  be the diagonal subgroup consisting of positive diagonals in both cases. Then  $M = \{e\}$  and  $M = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in [0, \pi) \right\}$  respectively.

Consider the following elements of  $\mathfrak{g}$ :

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We will need explicit form of ladder operators between the subspaces  $V_\ell^M$ 's.

**2.1. Ladder operators of  $\mathrm{PSL}_2(\mathbb{R})$ .** Set

$$\mathcal{R} = \frac{1}{2}(\pi(H) + i\pi(E + F)) \quad \text{and} \quad \overline{\mathcal{R}} = \frac{1}{2}(\pi(H) - i\pi(E + F)).$$

It is well-known that  $\mathcal{R}(V_\ell) = V_{\ell+1}$  and  $\overline{\mathcal{R}}(V_\ell) = V_{\ell-1}$  for any  $\ell \in \mathbb{Z}$  (cf. [1, Prop.2.5.2]).

We set

$$v_\ell = \begin{cases} \mathcal{R}^\ell(v_0) & \text{if } \ell \geq 0 \\ \overline{\mathcal{R}}^{|\ell|}(v_0) & \text{if } \ell < 0 \end{cases}$$

We write  $\|v\|$  for the norm of  $v \in V$ :  $\|v\| = \sqrt{\langle v, v \rangle}$ . Recall that  $v_0$  is a unit vector in  $V^K$ .

For  $\ell \geq 0$ , we have (see [5]):

$$\|v_{\pm\ell}\|_2 = \frac{\sqrt{\Gamma(s+\ell)\Gamma(1-s+\ell)}}{\sqrt{\Gamma(s)\Gamma(1-s)}}. \quad (2.1)$$

where  $\Gamma(x)$  denotes the Gamma function.

**2.2. Ladder operators of  $\mathrm{PSL}_2(\mathbb{C})$ .** For  $\mathrm{PSL}_2(\mathbb{C})$ , set

$$\begin{aligned} E^+ &= \frac{1}{2}(\pi(E - F) - i\pi(i(E + F))), \quad E^- = \frac{1}{2}(-\pi(E - F) - i\pi(i(E + F))), \\ R &= \frac{1}{2}(\pi(E + F) - i\pi(i(E - F))), \quad L = \frac{1}{2}(-\pi(E + F) - i\pi(i(E - F))), \\ D &= \frac{1}{2}\pi(iH) \quad \text{and} \quad \tilde{H} = \frac{1}{2}\pi(H). \end{aligned}$$

We remark that the operators  $E^\pm$  moves between different  $M$ -types inside each fixed  $V_\ell$  and  $R$  (resp.  $L$ ) moves the highest (resp. lowest)

weight vector space of each  $V_\ell$  into the highest (resp. lowest) weight vector space of  $V_{\ell+1}$ .

**Lemma 2.2.** *Let  $\ell \in \mathbb{Z}_{\geq 0}$ . We have*

(1)

$$\mathcal{C}_K \mathcal{Z}_\ell = \mathcal{Z}_\ell \mathcal{C}_K + i(RE^- D - LE^+ D) + \tilde{H}(-2\mathcal{C}_K + 2D^2) - 2(\ell+1)\mathcal{Z}_\ell - 2(\ell+1)\ell\tilde{H};$$

(2)  $D\mathcal{Z}_\ell = \mathcal{Z}_\ell D$ ;

(3)

$$\mathcal{Z}_\ell \tilde{H} = \tilde{H} \mathcal{Z}_\ell + \mathcal{C} + 2\mathcal{C}_K - D^2 - \tilde{H}^2.$$

*Proof.* We have

$$\mathcal{C}_K = D^2 - \frac{1}{2}(E^+ E^- + E^- E^+) \quad (2.3)$$

We compute

- (1)  $\mathcal{C}_K \tilde{H} = \tilde{H} \mathcal{C}_K + 2\mathcal{Z}_\ell + 2\ell\tilde{H}$ ,
- (2)  $\mathcal{C}_K R = R \mathcal{C}_K + 2iDR + 2\tilde{H}E^+$ , and
- (3)  $\mathcal{C}_K L = L \mathcal{C}_K - 2iDL + 2\tilde{H}E^-$ .

These relations imply

$$\begin{aligned} \mathcal{C}_K \mathcal{Z}_\ell &= \frac{1}{2} \left( RE^- + LE^+ - 2(\ell+1)\tilde{H} \right) \mathcal{C}_K \\ &\quad + \left\{ iDRE^- - iDLE^+ + \tilde{H}(E^+ E^- + E^- E^+) - 2(\ell+1)(\mathcal{Z}_\ell + \ell\tilde{H}) \right\}. \end{aligned}$$

Since

$$E^+ E^- = -\mathcal{C}_K + D^2 - iD, \quad E^- E^+ = -\mathcal{C}_K + D^2 + iD, \quad DRE^- = RE^- D$$

and

$$DLE^+ = LE^+ D,$$

we compute that  $\mathcal{C}_K \mathcal{Z}_\ell$  is equal to

$$\mathcal{Z}_\ell \mathcal{C}_K + i(RE^- D - LE^+ D) + \tilde{H}(-2\mathcal{C}_K + 2D^2) - 2(\ell+1)\mathcal{Z}_\ell - 2(\ell+1)\ell\tilde{H}.$$

For (2), we note that  $[D, R] = iR$ ,  $[D, L] = -iL$ ,  $[\tilde{H}, D] = 0$ ,  $[D, E^\pm] = \pm iE^\pm$ , and hence

$$\begin{aligned} D\mathcal{Z}_\ell &= \frac{1}{2} \left( DRE^- + DLE^+ - 2(\ell+1)D\tilde{H} \right) \\ &= \frac{1}{2} \left\{ (iR + RD)E^- + (-iL + LD)E^+ - 2(\ell+1)\tilde{H}D \right\} \\ &= \frac{1}{2} \left\{ RE^- + LE^+ - 2(\ell+1)\tilde{H} \right\} D = \mathcal{Z}_\ell D. \end{aligned}$$

(3) can be proved similarly using that the Casimir operator  $\mathcal{C}$  is given by

$$\mathcal{C} = \frac{1}{2}(2\tilde{H}^2 - 2D^2 - RL - LR + E^+ E^- + E^- E^+). \quad (2.4)$$

□

We show that the following differential operator  $\mathcal{Z}_\ell$  moves  $V_\ell^M$  to  $V_{\ell+1}^M$ : for each  $\ell \geq 0$ , set

$$\mathcal{Z}_\ell := \frac{1}{2} \left( RE^- + LE^+ - 2(\ell + 1)\tilde{H} \right).$$

**Proposition 2.5.** *For each  $\ell \geq 0$ ,*

$$\mathcal{Z}_\ell(V_\ell^M) \subset V_{\ell+1}^M.$$

*Proof.* Recall that  $\mathcal{C}_K$  acts on each  $V_\ell$  as a scalar multiplication  $-\ell(\ell + 1)$ . Hence  $v \in V_\ell$  if and only if  $\mathcal{C}_K v = -\ell(\ell + 1)v$ , and  $v \in V^M$  if and only if  $Dv = 0$ .

Let  $v \in V_\ell^M$ . Using  $Dv = 0$ , we deduce, using Lemma 2.2 (1) that

$$\begin{aligned} \mathcal{C}_K(\mathcal{Z}_\ell v) &= -\ell(\ell + 1)\mathcal{Z}_\ell v + 2\ell(\ell + 1)\tilde{H}v - 2(\ell + 1)\mathcal{Z}_\ell v - 2\ell(\ell + 1)\tilde{H}v \\ &= -(\ell + 1)(\ell + 2)\mathcal{Z}_\ell v. \end{aligned}$$

Hence  $\mathcal{Z}_\ell v \in V_{\ell+1}$ . By Lemma 2.2 (2), we have

$$D\mathcal{Z}_\ell v = \mathcal{Z}_\ell(Dv) = 0,$$

finishing the proof. □

**Theorem 2.6** (Recursive formula for  $v_\ell$ ). *Fixing a unit vector  $v_0 \in V^K$ , define  $v_\ell$  recursively:*

$$v_\ell := \mathcal{Z}_\ell(v_{\ell-1}).$$

*Then  $v_1 = -\tilde{H}v_0$ , and for  $\ell \geq 2$ ,*

$$v_\ell = (-2\ell + 1)\tilde{H}v_{\ell-1} + (\ell - 1)^2(\ell(\ell - 2) - s(s - 2))v_{\ell-2}.$$

*Proof.* Note that  $\mathcal{C}$  acts on  $V$  by  $\lambda = -s(n - 1 - s)$ . For  $\ell \geq 0$  and  $m \geq 0$ , we have

$$\begin{aligned} \mathcal{Z}_\ell &= \frac{1}{2}(RE^- + LE^+ - 2(\ell + 1)\tilde{H}) \\ &= \frac{1}{2}(RE^- + LE^+ - 2(m + 1)\tilde{H} + (-2(\ell + 1) + 2(m + 1))\tilde{H}) \\ &= \mathcal{Z}_m + (m - \ell)\tilde{H} \end{aligned}$$

and hence

$$\mathcal{Z}_\ell v_m = v_{m+1} + (m - \ell)\tilde{H}v_m.$$

Since  $E^+v_0 = E^-v_0 = 0$ ,

$$v_1 = \mathcal{Z}_0 v_0 = -\tilde{H}v_0.$$

We claim that for  $\ell \geq 2$ ,

$$v_\ell = a_\ell \tilde{H}v_{\ell-1} + b_\ell v_{\ell-2}$$

where

$$a_\ell = -2\ell + 1 \text{ and } b_\ell = (\ell - 1)^2(\ell(\ell - 2) - s(s - 2))$$

For the induction process, assume  $v_\ell = a_\ell \tilde{H}v_{\ell-1} + b_\ell v_{\ell-2}$ . We deduce that, using Lemma 2.2 (3),

$$\begin{aligned} v_{\ell+1} &= \mathcal{Z}_\ell v_\ell \\ &= a_\ell(\mathcal{Z}_\ell \tilde{H}v_{\ell-1}) + b_\ell(\mathcal{Z}_\ell v_{\ell-2}) \\ &= a_\ell(\tilde{H}\mathcal{Z}_\ell + \mathcal{C} + 2\mathcal{C}_K - D^2 - \tilde{H}^2)v_{\ell-1} + b_\ell(v_{\ell-1} - 2\tilde{H}v_{\ell-2}). \end{aligned}$$

Using the induction hypothesis, as well as  $\mathcal{C}v_{\ell-1} = \lambda v_{\ell-1}$ ,  $\mathcal{C}_K v_{\ell-1} = -\ell(\ell - 1)v_{\ell-1}$  and  $Dv_{\ell-1} = 0$ , we have

$$\begin{aligned} v_{\ell+1} &= a_\ell(\tilde{H}(v_\ell - \tilde{H}v_{\ell-1}) + (\lambda - 2\ell(\ell - 1))v_{\ell-1} - \tilde{H}^2 v_{\ell-1}) \\ &\quad + b_\ell(v_{\ell-1} - 2\tilde{H}v_{\ell-2}) \\ &= a_\ell \tilde{H}v_\ell - 2\tilde{H}(a_\ell \tilde{H}v_{\ell-1} + b_\ell v_{\ell-2}) + (a_\ell(\lambda - 2\ell(\ell - 1)) + b_\ell)v_{\ell-1} \\ &= (a_\ell - 2)\tilde{H}v_\ell + (a_\ell(\lambda - 2\ell(\ell - 1)) + b_\ell)v_{\ell-1} \\ &= a_{\ell+1} \tilde{H}v_\ell + b_{\ell+1} v_{\ell-1}. \end{aligned}$$

□

**Lemma 2.7.** *Let  $v_\ell$  be as in Theorem 2.6. For each  $\ell \geq 0$ ,*

$$\|v_\ell\| = \frac{\ell!}{\sqrt{2\ell+1}} \frac{\sqrt{\Gamma(s+\ell)\Gamma(2-s+\ell)}}{\sqrt{\Gamma(s)\Gamma(2-s)}}.$$

*In particular,  $v_\ell \neq 0$  for each  $\ell \geq 0$ .*

*Proof.* Note that  $v_\ell$ 's are mutually orthogonal to each other. By Lemma 2.6, for  $\ell \geq 2$ , we have  $v_\ell = a_\ell \tilde{H}v_{\ell-1} + b_\ell v_{\ell-2}$  where  $a_\ell = -2\ell + 1$  and  $b_\ell = (\ell - 1)^2(\ell(\ell - 2) - s(s - 2))$ . Therefore

$$\begin{aligned} \|v_\ell\|^2 &= \langle a_\ell \tilde{H}v_{\ell-1} + b_\ell v_{\ell-2}, v_\ell \rangle = \langle a_\ell \tilde{H}v_{\ell-1}, v_\ell \rangle \\ &= -\langle a_\ell v_{\ell-1}, \tilde{H}v_\ell \rangle = -\left\langle a_\ell v_{\ell-1}, \frac{1}{a_{\ell+1}}v_{\ell+1} - \frac{b_{\ell+1}}{a_{\ell+1}}v_{\ell-1} \right\rangle \\ &= \frac{a_\ell b_{\ell+1}}{a_{\ell+1}} \|v_{\ell-1}\|^2. \end{aligned}$$

It follows that

$$\begin{aligned}
\|v_\ell\|^2 &= \frac{a_\ell b_{\ell+1}}{a_{\ell+1}} \cdot \frac{a_{\ell-1} b_\ell}{a_\ell} \cdot \dots \cdot \frac{a_{\ell+1-j} b_{\ell+2-j}}{a_{\ell+2-j}} \dots \frac{a_1 b_2}{a_2} \|v_0\|^2 \\
&= \frac{a_1}{a_{\ell+1}} \cdot \prod_{j=1}^{\ell} b_{j+1} \\
&= \frac{1}{2\ell+1} \prod_{j=1}^{\ell} j^2 (j^2 - 1 - s(s-2)) \\
&= \frac{(\ell!)^2}{2\ell+1} \prod_{j=1}^{\ell} (j+s-1)(j-s+1).
\end{aligned}$$

Therefore the claim follows by the well-known identity on the Gamma function.  $\square$

This lemma shows in particular that  $\mathcal{Z}_\ell$  maps a non-zero vector to a non-zero vector. Hence we state:

**Corollary 2.8.** *We have for each  $\ell \in \mathbb{Z}_{\geq 0}$ ,*

$$\mathcal{Z}_\ell(V_\ell^M) = V_{\ell+1}^M \quad \text{and} \quad V^M = \oplus_{\ell \in \hat{K}} \mathbb{C} v_\ell.$$

### 3. EXPLICIT FORMULAS FOR BASE EIGENFUNCTIONS $\phi_\ell$

Let  $G = \mathrm{PSL}_2(\mathbb{R})$  or  $\mathrm{PSL}_2(\mathbb{C})$ . Consider the upper half-space:  $\mathbb{H}^n = \{(x, y) : x \in \mathbb{R}^{n-1}, y > 0\}$  and set  $j = (0_{n-1}, 1)$ . We set  $K = \mathrm{Stab}_G(j)$ . So  $K = \mathrm{PSO}(2)$  and  $\mathrm{PSU}(2)$  respectively. The geometric boundary of  $\mathbb{H}^n$  is naturally identified with  $\mathbb{R}^n \cup \{\infty\}$ , and  $G = \mathrm{Isom}^+(\mathbb{H}^n)$ .

We set  $N$  to be the strict upper triangular subgroup of  $G$  and  $A$  the diagonal subgroup consisting of positive diagonals. We have the Iwasawa decomposition  $G = NAK$ : any element of  $g$  is written uniquely as  $n_x a_y k$  where  $n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in N$ ,  $a_y = \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \sqrt{y}^{-1} \end{pmatrix} \in A$  and  $k \in K$ .

Let  $\Gamma < G$  be a geometrically finite discrete subgroup with critical exponent  $\frac{n-1}{2} < \delta < n-1$ . Let  $\nu_j$  denote the Patterson-Sullivan measure on the limit set  $\Lambda(\Gamma)$  with respect to the basepoint  $j \in \mathbb{H}^n$ . Up to a scaling,  $\nu_j$  is the weak-limit as  $s \rightarrow \delta^+$  of the family of measures

$$\nu_j(s) := \frac{1}{\sum_{\gamma \in \Gamma} e^{-sd(j, \gamma j)}} \sum_{\gamma \in \Gamma} e^{-sd(j, \gamma j)} \delta_{\gamma(j)}$$

where  $\delta_{\gamma(j)}$  is the dirac measure at  $\gamma(j)$ .

For  $n = 2, 3$ , the Laplacian operators  $\Delta$  on  $\mathbb{H}^n$  are respectively given by

$$\Delta = -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \quad \text{and} \quad \Delta = -y^2 \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial y^2} \right) + y \frac{\partial}{\partial y}.$$

The Casimir operator  $\mathcal{C}$  given in section (2) satisfies  $\mathcal{C}(\psi) = -\Delta(\psi)$  for all  $\psi \in C^\infty(\Gamma \backslash G)^K$  via the identification of  $G/K$  with  $\mathbb{H}^n$ .

We consider the Hilbert space  $L^2(\Gamma \backslash G)$  where the inner product  $\langle \psi_1, \psi_2 \rangle$  is given by

$$\langle \psi_1, \psi_2 \rangle = \int_{\Gamma \backslash G} \psi_1(g) \overline{\psi_2(g)} dg$$

where  $dg$  denotes a  $G$ -invariant measure on  $\Gamma \backslash G$ .

The action of  $G$  on  $L^2(\Gamma \backslash G)$  by right translations is a unitary representation.

The following is a consequence of a theorem of Sullivan [35] and of Lax-Phillips [21]:

**Theorem 3.1.** *There exist complementary series representations  $\mathcal{V}_i$ ,  $i = 0, \dots, k$  of  $G$  such that  $-\mathcal{C}$  acts on  $\mathcal{V}_i^\infty$  by the scalar  $\alpha_i = s_i(n - 1 - s_i)$  where  $\alpha_0 = \delta(n - 1 - \delta) < \alpha_1 \leq \dots \leq \alpha_k < (n - 1)^2/4$  such that*

$$L^2(\Gamma \backslash G) = \mathcal{V}_0 \oplus \dots \oplus \mathcal{V}_k \oplus \mathcal{W}$$

*and that  $\mathcal{W}$  consists of tempered spectrum (i.e., principal series and discrete series).*

We set  $V := \mathcal{V}_0$ . The base eigenfunction  $\phi_0 \in V^K$  for the Laplacian  $\Delta$  has eigenvalue  $\delta(n - 1 - \delta)$  and can be explicitly written as the integral of the Poisson kernel against  $\nu_j$ :

$$\phi_0(n_x a_y) = \int_{u \in \mathbb{R}^n} \hat{\phi}_u(x, y) d\nu_j(u)$$

where  $\hat{\phi}_u(x, y) := \left( \frac{(|u|^2 + 1)y}{|x - u|^2 + y^2} \right)^\delta$  by Sullivan [35]. We normalize  $\nu_j$  so that  $\|\phi_0\|_2 = 1$ .

For  $n = 2$ , we set

$$\psi_\ell = \begin{cases} \mathcal{R}^\ell(\phi_0) & \text{if } \ell \geq 0 \\ \overline{\mathcal{R}^{|\ell|}}(\phi_0) & \text{if } \ell < 0 \end{cases}$$

Sine  $\overline{\mathcal{R}}$  is the complex conjugate of  $\mathcal{R}$ ,  $\psi_{-\ell} = \overline{\psi_\ell}$ .

For  $n = 3$ , we define  $\psi_\ell$  recursively:

$$\psi_0 := \phi_0, \quad \text{and} \quad \psi_\ell = \mathcal{Z}_\ell(\psi_{\ell-1}) \quad \text{for each } \ell \geq 1.$$

**Definition 3.2.** For each  $\ell \in \hat{K}$ , define the unit vector in  $V_\ell^M$  by:

$$\phi_\ell := \frac{\psi_\ell}{\|\psi_\ell\|_2} \in C^\infty(\Gamma \backslash G)^M \cap L^2(\Gamma \backslash G).$$

By Corollary 2.8,  $\psi_\ell$ 's are non-zero vectors, and hence  $\phi_\ell$ 's are well-defined and that

$$V^M = \oplus_{\ell \in \hat{K}} \mathbb{C} \phi_\ell.$$

### 3.1. Base eigenfunctions for $G = \mathrm{PSL}_2(\mathbb{R})$ .

**Theorem 3.3.** Let  $G = \mathrm{PSL}_2(\mathbb{R})$ . Let  $\ell \in \mathbb{Z}_{\geq 0}$ .

$$\phi_\ell(n_x a_y) = \frac{\sqrt{\Gamma(1-\delta)\Gamma(\ell+\delta)}}{\sqrt{\Gamma(\delta)\Gamma(\ell+1-\delta)}} \cdot \int_{\mathbb{R}} \hat{\phi}_u(x, y) \left( \frac{(x-u)-iy}{(x-u)+iy} \right)^\ell d\nu_j(u).$$

In particular,

$$|\phi_{\pm\ell}(n_x a_y)| \ll \phi_0(n_x a_y)$$

with implied constant independent of  $\ell$ .

*Proof.* Letting  $k_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$ , we have  $K = \{k_\theta : \theta \in [0, \pi)\}$ .

The raising operator  $\mathcal{R}$  can be written as

$$\mathcal{R} = e^{2i\theta} \left( iy \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + \frac{1}{2i} \frac{\partial}{\partial \theta} \right).$$

By (2.1), it suffices to show that

$$\psi_\ell(n_x a_y k_\theta) = \frac{e^{2\ell i\theta} \Gamma(\delta+\ell)}{\Gamma(\delta)} \int_{\mathbb{R}} \hat{\phi}_u(x, y) \left( \frac{(x-u)-iy}{(x-u)+iy} \right)^\ell d\nu_j(u).$$

To use an induction, we assume that

$$(\mathcal{R}^\ell \phi_0)(n_x a_y k_\theta) = \frac{\Gamma(\delta+\ell)}{\Gamma(\delta)} \cdot e^{2i\ell\theta} \int_{\mathbb{R}} \hat{\phi}_u(x, y) \left( \frac{(x-u)-iy}{(x-u)+iy} \right)^\ell d\nu_j(u).$$

We compute

$$\begin{aligned} & \left( iy \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) \left( \hat{\phi}_u(x, y) \cdot \left( \frac{(x-u)-iy}{(x-u)+iy} \right)^\ell \right) \\ &= \delta \cdot \hat{\phi}_u(x, y) \cdot \left( \frac{(x-u)-iy}{(x-u)+iy} \right)^{\ell+1} + \ell \cdot \hat{\phi}_u(x, y) \cdot \left( \frac{(x-u)-iy}{(x-u)+iy} \right)^\ell \cdot \left( \frac{-2iy}{(x-u)+iy} \right) \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{2i} \frac{\partial}{\partial \theta} (\mathcal{R}^\ell \phi_0)(n_x a_y k_\theta) \\ &= \frac{\Gamma(\delta+\ell)}{\Gamma(\delta)} \cdot \ell \cdot e^{2i\ell\theta} \int_{\mathbb{R}} \hat{\phi}_u(x, y) \left( \frac{(x-u)-iy}{(x-u)+iy} \right)^\ell d\nu_j(u). \end{aligned}$$



Hence

$$(\mathcal{R}^{\ell+1}\phi_0)(n_x a_y k_\theta) = \frac{\Gamma(\delta+\ell+1)}{\Gamma(\delta)} \cdot e^{2i(\ell+1)\theta} \int_{\mathbb{R}} \hat{\phi}_u(x, y) \left( \frac{(x-u)-iy}{(x-u)+iy} \right)^{\ell+1} d\nu_j(u).$$

□

**3.2. Base eigenfunctions for  $G = \text{PSL}_2(\mathbb{C})$ .** We parameterize elements of  $K = \text{PSU}(2)$  as

$$K = \left\{ k_{\mu_1, \mu_2, \theta} : \begin{array}{l} 0 \leq \theta < \frac{\pi}{2}, \ 0 \leq \mu_1 < \pi, \ 0 \leq \mu_2 < 2\pi, \\ \mu_1 = 0 \text{ if } \theta = \frac{\pi}{2} \text{ and } \mu_2 = 0 \text{ if } \theta = 0 \end{array} \right\}$$

where

$$k_{\mu_1, \mu_2, \theta} := \begin{pmatrix} e^{i\mu_1} \cos \theta & e^{i\mu_2} \sin \theta \\ -e^{-i\mu_2} \sin \theta & e^{-i\mu_1} \cos \theta \end{pmatrix}.$$

The differential operator  $\tilde{H}$  in section 2 in terms of  $(x_1, x_2, y, \mu_1, \mu_2, \theta)$  coordinates, which can be obtained by direct computations:

$$\begin{aligned} \tilde{H} = & -\cos(\mu_1 + \mu_2) \sin(2\theta) y \frac{\partial}{\partial x_1} - \sin(\mu_1 + \mu_2) \sin(2\theta) y \frac{\partial}{\partial x_2} \\ & + \cos(2\theta) y \frac{\partial}{\partial y} + \frac{\sin(2\theta)}{2} \frac{\partial}{\partial \theta}. \end{aligned} \quad (3.4)$$

Let  $\mathbf{P}_\ell(t)$  denote the Legendre polynomial of degree  $\ell$ . It is defined by the recursive relation:  $\mathbf{P}_0(t) = 1$ ,  $\mathbf{P}_1(t) = t$  and

$$\ell \cdot \mathbf{P}_\ell(t) = (2\ell - 1)t\mathbf{P}_{\ell-1}(t) - (\ell - 1)^2\mathbf{P}_{\ell-2}(t).$$

**Theorem 3.5.** *Let  $\ell \geq 1$ .*

$$\phi_\ell(n_x a_y) = \sqrt{2\ell + 1} \cdot \frac{\sqrt{\Gamma(2-\delta)\Gamma(\ell+\delta)}}{\sqrt{\Gamma(\delta)\Gamma(\ell+2-\delta)}} \cdot \int_{u \in \mathbb{R}^2} \hat{\phi}_u(x, y) \cdot \mathbf{P}_\ell \left( \frac{y^2 - |x-u|^2}{y^2 + |x-u|^2} \right) d\nu_j(u).$$

In particular,

$$|\phi_\ell(n_x a_y)| \ll \sqrt{\ell + 1} \cdot \phi_0(n_x a_y)$$

where the implied constant is independent of  $\ell$ .

*Proof.* By Lemma 2.7, it suffices to show that

$$\psi_\ell(n_x a_y) = \ell! \frac{\Gamma(\delta+\ell)}{\Gamma(\delta)} \int_{u \in \mathbb{R}^2} \hat{\phi}_u(x, y) \cdot \mathbf{P}_\ell \left( \frac{y^2 - |x-u|^2}{y^2 + |x-u|^2} \right) d\nu_j(u). \quad (3.6)$$

Recall that  $\psi_0 = \phi_0$ ,  $\psi_1 = -\tilde{H}\psi_0$ , and for  $\ell \geq 1$ ,

$$\psi_{\ell+1} = a_\ell \tilde{H}(\psi_\ell) + b_\ell \psi_{\ell-1}$$

where  $a_\ell = -2\ell + 1$  and  $b_\ell = (\ell - 1)^2(\delta(2 - \delta) + \ell(\ell - 2))$ .

Setting  $\hat{\phi}_u(n_x a_y k) = \hat{\phi}_u(x, y)$ , we also define functions  $\hat{\phi}_u^{(\ell)}$  on  $G$  by the recursive formula:  $\hat{\phi}_u^{(1)}(n_x a_y k) := -\tilde{H}\hat{\phi}_u(n_x a_y k)$  and for  $\ell \geq 2$ ,  $\hat{\phi}_u^{(\ell)}(n_x a_y k) := a_\ell \tilde{H}(\hat{\phi}_u^{(\ell-1)})(n_x a_y k) + b_\ell \hat{\phi}_u^{(\ell-2)}(n_x a_y k)$ .

Since all other terms in  $\tilde{H}$  besides the term  $y \frac{\partial}{\partial y}$  have  $\sin(2\theta)$  and hence vanish when  $\theta = 0$ , we have:

$$\hat{\phi}_u^{(\ell)}(x, y, e) = a_\ell y \frac{\partial}{\partial y} \hat{\phi}_u^{(\ell-1)}(x, y, e) + b_\ell \hat{\phi}_u^{(\ell-2)}(x, y, e). \quad (3.7)$$

Since  $\psi_\ell(n_x a_y) = \int_{u \in \Lambda(\Gamma)} \hat{\phi}_u^{(\ell)}(x, y, e) d\nu_j(u)$  by (3.7), (3.6) follows if we show:

$$\hat{\phi}_u^{(\ell)}(x, y, e) = \frac{\Gamma(\delta+\ell)}{\Gamma(\delta)} \cdot \hat{\phi}_u(x, y) \cdot \ell! \cdot \mathbf{P}_\ell(B(x, y, u)) \quad (3.8)$$

where  $B(x, y, u) = \frac{y^2 - |x-u|^2}{y^2 + |x-u|^2}$ .

Since

$$y \frac{\partial}{\partial y} \hat{\phi}_u(x, y) = -\delta \cdot \hat{\phi}_u(x, y) \cdot B(x, y, u) \quad \text{and} \quad y \frac{\partial}{\partial y} B = -B^2 + 1,$$

we have

$$\begin{aligned} \hat{\phi}_u^{(1)}(x, y, e) &= a_1 \left( y \frac{\partial}{\partial y} \hat{\phi}_u \right) (x, y, e) \\ &= -\delta \hat{\phi}_u(x, y) \cdot a_1 B = \delta \hat{\phi}_u(x, y) \mathbf{P}_1(B). \end{aligned}$$

For  $j \leq \ell - 1$ , assume that  $\hat{\phi}_u^{(j)}(x, y, e) = d_j \hat{\phi}_u(x, y) j! \cdot \mathbf{P}_j(B)$  where  $d_\ell := \prod_{j=1}^{\ell} (\delta + j - 1)$ . Then

$$\begin{aligned} \hat{\phi}_u^{(\ell)}(x, y, e) &= a_\ell \cdot \left( y \frac{\partial}{\partial y} \hat{\phi}_u^{(\ell-1)} \right) (x, y, e) + b_\ell \cdot \hat{\phi}_u^{(\ell-2)}(x, y, e) \\ &= a_\ell d_{\ell-1} \cdot y \frac{\partial}{\partial y} \left( \hat{\phi}_u(x, y) \cdot (\ell-1)! \mathbf{P}_{\ell-1}(B) \right) + b_\ell d_{\ell-2} \cdot \hat{\phi}_u(x, y) \cdot (\ell-2)! \cdot \mathbf{P}_{\ell-2}(B). \end{aligned}$$

Since  $b_\ell = (\ell-1)^2(-\delta+\ell)(\delta+\ell-2)$ , we have

$$d_{\ell-2} b_\ell = (\ell-1)^2 d_{\ell-1} (-\delta + \ell).$$

Then

$$\begin{aligned} \hat{\phi}_u^{(\ell)}(x, y, e) &= d_{\ell-1} \left\{ -a_\ell \delta B(\ell-1)! \mathbf{P}_{\ell-1}(B) - a_\ell (B^2 - 1)(\ell-1)! \mathbf{P}'_{\ell-1}(B) \right. \\ &\quad \left. + (\ell-1)^2 (-\delta + \ell)(\ell-2)! \mathbf{P}_{\ell-2}(B) \right\} \hat{\phi}_u(x, y) \\ &= d_{\ell-1} \left\{ -a_\ell \delta B(\ell-1)! \mathbf{P}_{\ell-1}(B) - \delta (\ell-1)^2 (\ell-2)! \mathbf{P}_{\ell-2}(B) \right. \\ &\quad \left. - a_\ell (B^2 - 1)(\ell-1)! \mathbf{P}'_{\ell-1}(B) + \ell (\ell-1)^2 (\ell-2)! \mathbf{P}_{\ell-2}(B) \right\} \hat{\phi}_u(x, y). \end{aligned}$$

Since

$$(\ell-1)(\ell-1)! \mathbf{P}_\ell(B) = -a_\ell (B^2 - 1)(\ell-1)! \mathbf{P}'_{\ell-1}(B) + (\ell-1)^2 \ell (\ell-2)! \mathbf{P}_{\ell-2}(B)$$

we have

$$\begin{aligned}\hat{\phi}_u^{(\ell)}(x, y, e) &= d_{\ell-1} \hat{\phi}_u(x, y) \cdot (\delta + \ell - 1) \ell! \mathbf{P}_\ell(B) \\ &= d_\ell \cdot \hat{\phi}_u(x, y) \cdot \ell! \mathbf{P}_\ell(B).\end{aligned}$$

This proves the first claim, using the relation  $\Gamma(x+1) = x\Gamma(x)$  and  $d_\ell = \frac{\Gamma(\delta+\ell)}{\Gamma(\delta)}$ . Since  $|\mathbf{P}_\ell(t)| \leq 1$  for  $|t| \leq 1$  (see p. 987 of [16]), the second claim follows.  $\square$

**Theorem 3.9.** *For each  $\ell \geq 0$ , there exists a constant  $C_\ell > 0$  such that for any  $n_x a_y k \in NAK$ ,*

$$|\phi_\ell(n_x a_y k)| \ll C_\ell \cdot \phi_0(n_x a_y).$$

*Proof.* We use notations from the proof of Theorem 3.5. We compute

$$y \frac{\partial}{\partial x_1} \hat{\phi}_u = \delta \cdot \hat{\phi}_u \cdot \frac{-2y(x_1 - u_1)}{|x - u|^2 + y^2}, \quad y \frac{\partial}{\partial x_2} \hat{\phi}_u = \delta \cdot \hat{\phi}_u \cdot \frac{-2y(x_2 - u_2)}{|x - u|^2 + y^2}$$

and

$$y \frac{\partial}{\partial y} \hat{\phi}_u = \delta \cdot \hat{\phi}_u \cdot \frac{|x - u|^2 - y^2}{|x - u|^2 + y^2}.$$

Let

$$A_1(x, y, u) = \frac{-2y(x_1 - u_1)}{|x - u|^2 + y^2}, \quad A_2(x, y, u) = \frac{-2y(x_2 - u_2)}{|x - u|^2 + y^2}$$

and

$$B(x, y, u) = \frac{|x - u|^2 - y^2}{|x - u|^2 + y^2}.$$

For  $\mu = (\mu_1, \mu_2)$  and  $0 \leq \theta < \pi$ , define

$$\Phi_1(\mu, \theta) := -\sin(2\theta) \cos(\mu_1 + \mu_2), \quad \text{and} \quad \Phi_2(\mu, \theta) := -\sin(2\theta) \sin(\mu_1 + \mu_2)$$

and  $\Psi(\theta) := \cos(2\theta)$  so that by (3.4),

$$\tilde{H} = \Phi_1 y \frac{\partial}{\partial x_1} + \Phi_2 y \frac{\partial}{\partial x_2} + \Psi y \frac{\partial}{\partial y} + \frac{\sin(2\theta)}{2} \frac{\partial}{\partial \theta}.$$

Hence

$$\tilde{H} \hat{\phi}_u = \delta \cdot \hat{\phi}_u \cdot (\Phi_1 A_1 + \Phi_2 A_2 + \Psi B);$$

so

$$\hat{\phi}_u^{(1)} = \hat{\phi}_u \cdot a_1 \delta \cdot (\Phi_1 A_1 + \Phi_2 A_2 + \Psi B).$$

We compute

$$y \frac{\partial}{\partial x_1} A_1 = A_1^2 + B - 1, \quad y \frac{\partial}{\partial x_2} A_1 = A_1 A_2 + B - 1, \quad y \frac{\partial}{\partial y} A_1 = A_1 B;$$

$$y \frac{\partial}{\partial x_1} A_2 = A_1 A_2 + B - 1, \quad y \frac{\partial}{\partial x_2} A_2 = A_2^2 + B - 1, \quad y \frac{\partial}{\partial y} A_2 = A_2 B;$$

and

$$y \frac{\partial}{\partial x_1} B = A_1(B - 1), \quad y \frac{\partial}{\partial x_2} B = A_2(B - 1), \quad y \frac{\partial}{\partial y} B = (B - 1)(B + 1).$$

We also compute:

$$\tilde{H}(\Phi_1) = \Phi_1 \Psi, \quad \tilde{H}(\Phi_2) = \Phi_2 \Psi$$

and

$$\tilde{H}(\Psi) = -\Phi_1^2 - \Phi_2^2 = -1 + \Psi^2.$$

It follows that  $\hat{\phi}_u^{(\ell)} = \hat{\phi}_u \cdot p_\ell(\Phi_1, \Phi_2, A_1, A_2, \Psi, B)$  where  $p_\ell$  is a polynomial in  $\Phi_1, \Phi_2, A_1, A_2, \Psi$  and  $B$ , whose coefficients are given by monomials in  $\{\pm 1, a_1, \dots, a_\ell, b_1, \dots, b_\ell, \delta\}$ . Since the absolute values of  $\Phi_1, \Phi_2, \Psi, A_1, A_2$  and  $B$  are all bounded above by 1, we deduce

$$\left| \hat{\phi}_u^{(\ell)}(x, y, k) \right| \leq C_\ell \cdot \hat{\phi}_u(x, y).$$

for some constant  $C_\ell > 0$  independent of  $x, y, k$ . Since

$$\psi_\ell(n_x a_y k) = \int_{u \in \Lambda(\Gamma)} \hat{\phi}_u^{(\ell)}(x, y, k) d\nu_j(u),$$

the claim follows.  $\square$

#### 4. HOROSPHERICAL AVERAGE OF $\phi_\ell$

We let  $G, \Gamma, \phi_0, \phi_\ell$ , etc., be as in Section 3. We assume that  $\Gamma \backslash \Gamma N$  is closed in  $\Gamma \backslash G$  in the whole section. The horosphere in  $T^1(\mathbb{H}^n)$  corresponding to  $NM/M$  is the upward normal vectors on the horizontal plane containing  $j$ , and hence based at  $\infty \in \partial(\mathbb{H}^n)$ .

The assumption that  $\Gamma \backslash \Gamma N$  is closed is equivalent to saying that either  $\infty \notin \Lambda(\Gamma)$  or  $\infty$  is a parabolic fixed point of  $\Gamma$  [8]. Recall  $\xi \in \Lambda(\Gamma)$  is a parabolic fixed point if it is a unique fixed point in  $\partial(\mathbb{H}^n)$  of an element of  $\Gamma$ . One of the important properties of a geometrically finite group  $\Gamma$  is that any parabolic fixed point  $\xi$  is bounded, meaning that the stabilizer  $\text{Stab}_\Gamma(\xi)$  acts cocompactly on  $\Lambda(\Gamma) - \{\xi\}$  [6].

Therefore we have in our setting:

**Lemma 4.1.**  *$N \cap \Gamma$  acts cocompactly on  $\Lambda(\Gamma) - \{\infty\}$ .*

As mentioned in the introduction, the rank of  $\infty$  is the rank of  $\Gamma \cap N$  as a free abelian group.

We will compute the horospherical average of  $\phi_\ell$  over  $(\Gamma \cap N) \backslash N$ .

**Definition 4.2.** *Given  $\psi \in C(\Gamma \backslash G)^M$  and  $g \in G$ , define  $\psi^N \in C(\Gamma \backslash G)^M$  by*

$$\psi^N(g) := \int_{n_x \in (N \cap \Gamma) \backslash N} \psi(n_x g) dx$$

where  $dx$  denotes the Lebesgue measure on  $\mathbb{R}^n$ , provided the integral converges.

**Proposition 4.3.** *There exists  $c_n(0) > 0$  such that*

$$\phi_0^N(a_y) = c_n(0)y^{n-1-\delta}$$

for all  $y > 0$ .

*Proof.* By [19], it was shown that  $\phi_0^N(a_y)$  converges absolutely and that there exists constants  $c_n(0) > 0$  and  $d_n(0) \in \mathbb{R}$  such that for all  $y > 0$

$$\phi_0^N(a_y) = c_n(0)y^{n-1-\delta} + d_n(0)y^\delta.$$

Since  $\phi_0 > 0$  and the above holds for all  $y > 0$ , it follows that  $d_n(0) \geq 0$ . We claim that  $d_n(0) = 0$ .

When  $\infty \notin \Lambda(\Gamma)$ , we can show by direct computations:

$$\phi_0^N(a_y) = \begin{cases} \frac{\sqrt{\pi}\Gamma(\delta-\frac{1}{2})}{\Gamma(\delta)} \cdot \int_{u \in \Lambda(\Gamma)} (|u|^2 + 1)^\delta d\nu_j(u) \cdot y^{1-\delta} & \text{if } n = 2 \\ \frac{\pi}{\delta-1} \cdot \int_{u \in \Lambda(\Gamma)} (|u|^2 + 1)^\delta d\nu_j(u) \cdot y^{2-\delta} & \text{if } n = 3 \end{cases} \quad (4.4)$$

(see [19]).

Now suppose  $\infty \notin \Lambda(\Gamma)$ .

As  $\Gamma$  is geometrically finite,  $\Gamma$  admits a polyhedron fundamental domain  $\mathcal{F}$  in  $\mathbb{H}^n$  such that  $F_0 \times [Y_0, \infty)$  injects to  $\mathcal{F}$  for some  $Y_0 \gg 1$  where  $F_0$  is a fundamental domain in  $\mathbb{R}^{n-1}$  for  $N \cap \Gamma$ . Let  $B_t = \{x \in F_0 : |x| < t\}$  for  $t > 1$ .

When  $\infty$  is a bounded parabolic fixed point of rank  $n-1$ , take  $t_0$  so that  $B_{t_0} = F_0$ , which is possible since  $F_0$  is bounded in this case. Then

$$\int_{x \in B_{t_0}} \phi_0(n_x a_y) dx \geq \frac{d_n(0)}{2} y^\delta. \quad (4.5)$$

Therefore using the Cauchy-Schwartz inequality,

$$\begin{aligned} \|\phi_0\|_2^2 &\geq \int_{Y_0}^\infty \int_{B_{t_0}} \phi_0(n_x a_y)^2 y^{-n} dx dy \\ &\geq \frac{1}{\text{vol}(B_{t_0})} \int_{Y_0}^\infty \left( \int_{B_{t_0}} \phi_0(n_x a_y) dx \right)^2 y^{-n} dy \\ &\geq \frac{d_2(0)^2}{4\text{vol}(B_{t_0})} \int_{Y_0}^\infty y^{2\delta-n} dy \end{aligned}$$

Since  $\delta > (n-1)/2$ ,  $\|\phi_0\|_2 = \infty$  unless  $d_n(0) = 0$ . Therefore  $d_n(0) = 0$ .

The remaining case is when  $n = 3$  and  $\infty$  is a bounded parabolic fixed point of rank one. In this case, it is shown in the proof of [19, Prop. 4.6] that for all sufficiently large  $t \gg 1$ , there exists  $d_t > 0$  such that  $b_t \rightarrow 0$  as  $t \rightarrow \infty$  and satisfies for all  $y > 0$ ,

$$\int_{x \in F_0 - B_t} \phi_0(n_x a_y) dx \leq b_t y^\delta.$$

Hence if  $d_3(0) > 0$ , then for some large  $t_0 > 0$ , we have

$$\int_{x \in B_{t_0}} \phi_0(n_x a_y) dx \geq \frac{d_3(0)}{2} y^\delta. \quad (4.6)$$

By repeating the same argument as in the previous case, this leads to a contradiction.  $\square$

**Lemma 4.7.** *Let  $y > 0$  and  $k \in K$ . We have*

$$\int_{(N \cap \Gamma) \backslash N} \frac{\partial}{\partial x_i} \phi_\ell(n_x a_y k) dx = 0.$$

*Proof.* By Theorems 3.3 and 3.9, we have

$$|\phi_\ell(n_x a_y k)| \leq C_\ell \cdot \phi_0(n_x a_y) \quad (4.8)$$

for some  $C_\ell > 0$ .

Suppose  $n = 3$ . If  $\infty \notin \Lambda(\Gamma)$  and hence  $N \cap \Gamma = \{e\}$ ,  $|\phi_0(n_x a_y k)| \rightarrow 0$  as  $|x| \rightarrow \infty$ . Hence, by (4.8),

$$|\phi_\ell(n_x a_y k)| \rightarrow 0.$$

Therefore

$$\begin{aligned} & \int_{x_1 \in \mathbb{R}} \frac{\partial}{\partial x_1} \phi_\ell(n_x a_y k) dx_1 \\ &= \lim_{t \rightarrow \infty} \int_{-t}^t \frac{\partial}{\partial x_1} \phi_\ell(n_x a_y k) dx_1 \\ &= \lim_{t \rightarrow \infty} (\phi_\ell(n_{t+ix_2} a_y k) - \phi_\ell(n_{(-t, x_2)} a_y k)) = 0. \end{aligned}$$

The other case of taking the partial derivative with respect to  $x_2$  is symmetric.

When  $\infty$  is a bounded parabolic fixed point of rank one, we may assume that  $N \cap \Gamma$  is  $n_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ , so that there exists a fundamental domain  $F_0$  in  $\mathbb{R}^{n-1}$  inside  $\{(x_1, x_2) : 0 \leq x_1 \leq 1\}$ . In this case,  $\phi_0(n_x a_y) \rightarrow 0$  as  $x_2 \rightarrow \infty$  [36]. Hence  $|\phi_\ell(n_x a_y k)| \rightarrow 0$ , as  $x_2 \rightarrow \infty$ . By a similar argument as above, this implies that

$$\int_{x_2 \in \mathbb{R}} \frac{\partial}{\partial x_2} \phi_\ell(n_x a_y k) dx_2 = 0.$$

On the other hand,

$$\int_{x_1 \in [0, 1]} \frac{\partial}{\partial x_1} \phi_\ell(n_x a_y k) dx_1 = \phi_\ell(n_1 n_{ix_2} a_y k) - \phi_\ell(n_{ix_2} a_y k) = 0$$

by the  $n_1 \in N \cap \Gamma$ -invariance of  $\phi_\ell$ .

If  $\infty$  has rank 2, we may assume that  $N \cap \Gamma$  contains  $n_1$  and  $n_i$ . Then the claim follows from Green's theorem and the invariance of  $\phi_\ell$  by  $N \cap \Gamma$  as in the last argument. The case  $n = 2$  can be shown similarly.  $\square$

For  $\ell \geq 0$ , define

$$M_\ell(\theta) := \ell! \cdot \mathbf{P}_\ell(-\cos 2\theta).$$

By the known properties of the Legendra polynomials  $\mathbf{P}_\ell$ 's, we can verify:

**Lemma 4.9.** (1)  $M_0(\theta) = 1$ ,  $M_1(\theta) = -\cos(2\theta)$ , and for  $\ell \geq 2$ ,

$$2(\ell + 1)M_\ell(\theta) = (-2\ell + 1)\cos(2\theta)M_{\ell-1}(\theta) - (\ell - 1)^2M_{\ell-2}(\theta).$$

(2) For each  $\ell \geq 2$ ,

$$\begin{aligned} 2(\ell + 1)M_\ell(\theta) &= 4(-2\ell + 1)\cos(2\theta)M_{\ell-1}(\theta) \\ &\quad + (-2\ell + 1)\sin(2\theta)M'_{\ell-1}(\theta) + 2(\ell - 1)^2(\ell - 2)M_{\ell-2}(\theta). \end{aligned}$$

The following theorem implies in particular that  $\phi_\ell^N(a_y)$  converges, which is a priori unclear as  $(N \cap \Gamma) \setminus N$  is unbounded in general.

**Theorem 4.10.** For  $\ell \geq 0$ , we have

$$\phi_\ell^N(a_y) = (-1)^{(n-2)\ell} \cdot c_n(0) \cdot \frac{\sqrt{\Gamma(n-1-\delta)\Gamma(\ell+\delta)}}{\sqrt{\Gamma(\delta)\Gamma(\ell+n-1-\delta)}} \cdot \sqrt{2(n-2)\ell+1} \cdot y^{n-1-\delta}.$$

In particular,

$$|\phi_\ell^N(a_y)| \ll \ell^{(n-2)/2} \cdot y^{n-1-\delta}$$

with the implied constant independent of  $\ell \gg 1$ .

*Proof.* By Theorems 3.3 and 3.5, we have

$$\begin{aligned} \left| \int_{(N \cap \Gamma) \setminus N} \phi_\ell(n_x a_y) dx \right| &\leq \int_{(N \cap \Gamma) \setminus N} |\phi_\ell(n_x a_y)| dx \\ &\ll \int_{(N \cap \Gamma) \setminus N} \phi_0(n_x a_y) dx. \end{aligned}$$

Hence by Proposition 4.3, the integral  $\phi_\ell^N(a_y)$  converges absolutely.

Let  $n = 2$ . To use an induction, we assume the following

$$\psi_\ell^N(a_y k_\theta) = e^{2\ell i \theta} c_2(0) \frac{\Gamma(\ell+1-\delta)}{\Gamma(1-\delta)} y^{1-\delta} \quad (4.11)$$

is true. Then applying Lemma 4.7,

$$\begin{aligned}
\psi_{\ell+1}^N(a_y k_\theta) &= e^{2i\theta} \int_{(N \cap \Gamma) \backslash N} \left( i y \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \frac{1}{2i} \frac{\partial}{\partial \theta} \right) \psi_\ell(n_x a_y k_\theta) dx \\
&= e^{2i\theta} \cdot \left( y \frac{\partial}{\partial y} \frac{1}{2i} \frac{\partial}{\partial \theta} \right) \psi_\ell^N(a_y k_\theta) \\
&= e^{2i\theta} \cdot \left( y \frac{\partial}{\partial y} \frac{1}{2i} \frac{\partial}{\partial \theta} \right) \left( e^{2\ell i\theta} c_2(0) \frac{\Gamma(\ell+1-\delta)}{\Gamma(1-\delta)} y^{1-\delta} \right) \\
&= e^{2(\ell+1)i\theta} \cdot \left( c_2(0) \frac{\Gamma(\ell+1-\delta)}{\Gamma(1-\delta)} \cdot ((1-\delta) + \ell) y^{1-\delta} \right) \\
&= e^{2(\ell+1)i\theta} \cdot \left( c_2(0) \frac{\Gamma(\ell+1+(1-\delta))}{\Gamma(1-\delta)} y^{1-\delta} \right)
\end{aligned}$$

since  $z\Gamma(z) = \Gamma(z+1)$ . This proves (4.11) for all positive integer  $\ell$ . Hence the claim follows from (2.1).

Let  $n = 3$ . Setting  $q_\ell := c_3(0) \prod_{j=1}^\ell (j+1-\delta)$ , we claim that

$$\psi_\ell^N(a_y k) = q_\ell \cdot y^{2-\delta} M_\ell(\theta) \quad (4.12)$$

where  $M_\ell(\theta)$  is defined as in Lemma 4.9. Set  $k = k_{\mu_1, \mu_2, \theta}$ ,  $a_\ell = -2\ell + 1$  and  $b_\ell = (\ell-1)^2(\delta(2-\delta) + \ell(\ell-2))$  for simplicity.

Since  $\phi_0$  is fixed by  $K$ ,  $E^\pm(\phi_0) = 0$ . Hence

$$\psi_1^N(a_y k) = a_1 \int_{(N \cap \Gamma) \backslash N} \tilde{H}(\phi_0)(n a_y k) dn.$$

By Proposition 4.3 and Lemma 4.7, using  $\frac{\partial}{\partial \theta} \phi_0(n_x a_y k) = 0$  and (3.4), we have

$$\begin{aligned}
\psi_1^N(a_y k) &= a_1 \left( \cos(2\theta) y \frac{\partial}{\partial y} + \frac{\sin(2\theta)}{2} \frac{\partial}{\partial \theta} \right) \psi_0^N(a_y k) \\
&= a_1 c_3(0) \cdot \cos(2\theta) (2-\delta) y^{2-\delta}
\end{aligned}$$

Hence (4.12) holds for  $\ell = 1$ .

Assume (4.12) for  $\ell$ , we deduce using Lemma 4.9 that

$$\begin{aligned}
\psi_{\ell+1}^N(a_y k) &= a_{\ell+1} q_\ell \tilde{H}(y^{2-\delta} M_\ell(\theta)) + b_{\ell+1} q_{\ell-1} y^{2-\delta} M_{\ell-1}(\theta) \\
&= (-2\ell-1) q_\ell (\cos(2\theta) (2-\delta) M_\ell(\theta) + \frac{\sin(2\theta)}{2} M'_\ell(\theta)) y^{2-\delta} \\
&\quad + \ell^2 (\ell+1-\delta) q_{\ell-1} (\delta + \ell - 1) y^{2-\delta} M_{\ell-1}(\theta)
\end{aligned}$$

By Lemma 4.9,

$$M_{\ell+1}(\theta) = (-2\ell-1) \cos(2\theta) M_\ell(\theta) - \ell^2 M_{\ell-1}(\theta)$$



and

$$(\ell + 2)M_{\ell+1}(\theta) = 2(-2\ell - 1)\cos(2\theta)M_\ell(\theta) + (-2\ell - 1)\frac{\sin(2\theta)}{2}M'_\ell(\theta) + \ell^2(\ell - 1)M_{\ell-1}(\theta).$$

Therefore

$$\psi_{\ell+1}^N(a_y k) = q_\ell(\ell + 2 - \delta)M_{\ell+1}(\theta) = q_{\ell+1}M_{\ell+1}(\theta),$$

proving (4.12). By dividing by the norm  $\|\psi_\ell\|_2$  given in Lemma 2.7, and using  $|\mathbf{P}_\ell(t)| \leq 1$  for all  $t \in [-1, 1]$  and  $\mathbf{P}_\ell(-1) = (-1)^\ell$ , we finish the proof.  $\square$

## 5. UNIFORM THICKENING FOR $\phi_\ell$ 'S

We continue the notations from section 4. and assume that  $\Gamma \setminus \Gamma N$  is closed in  $\Gamma \setminus G$ .

Fix a fundamental domain  $F_0$  for  $N \cap \Gamma$  in  $\mathbb{R}^{n-1}$  and choose a compact fundamental domain  $F_\Lambda \subset F_0$  for  $(N \cap \Gamma) \setminus \Lambda(\Gamma) - \{\infty\}$  given by Lemma 4.1.

**Lemma 5.1.** *Fix  $\ell \geq 0$ . Suppose  $(N \cap \Gamma) \setminus N$  is unbounded. For any open subset  $J \subset F_0$  containing  $F_\Lambda$ , we have for all  $0 < y < 1$ ,*

(1)

$$\int_{J^c} \phi_0(n_x a_y) dx \ll y^\delta;$$

(2)

$$\left| \int_{F_0 - J} \phi_\ell(n_x a_y) dx \right| \ll (\ell + 1)^{(n-2)/2} y^\delta$$

with the implied constant independent of  $\ell$ .

*Proof.* Let  $n = 2$ . Then  $\infty \notin \Lambda(\Gamma)$  and hence  $\epsilon_0 := \inf_{x \notin J, u \in \Lambda(\Gamma)} |x - u| > 0$ . Then by the change of variable  $w = \frac{x-u}{y}$  we have

$$\int_{F_0 - J} \phi_0(n_x a_y) dx \leq 2y^{1-\delta} \int_{u \in \Lambda(\Gamma)} (u^2 + 1)^\delta d\nu_i(u) \cdot \int_{w=\epsilon_0/y}^\infty \left( \frac{1}{w^2 + 1} \right)^\delta dw$$

The latter integral can be evaluated explicitly as an incomplete Beta function which has known asymptotics:

$$\int_{\epsilon_0/y}^\infty \left( \frac{1}{w^2 + 1} \right)^\delta dw = c \beta_{y^2/\epsilon_0^2}(\delta - 1/2, 1 - \delta),$$

where

$$\beta_z(\alpha, \beta) \ll z^\alpha.$$

Hence

$$\int_{F_0-J} \phi_0(n_x a_y) dx \ll y^{1-\delta} \cdot y^{2(\delta-1/2)} = y^\delta.$$

This proves (1) for  $n = 2$ , and we refer to [19, Proposition 3.7] for  $n = 3$ .

For the second claim, note that, by Theorem 3.5,

$$\left| \int_{x \in F_0-J} \phi_\ell(n_x a_y) dx \right| \leq \int_{x \in F_0-J} |\phi_\ell(n_x a_y)| dx \ll (|\ell|+1)^{(n-2)/2} \int_{F_0-J} \phi_0(n_x a_y) dx$$

since  $\phi_0$  is a positive function. Hence the claim (2) follows from (1).  $\square$

The associated Legendra function  $\mathbf{P}_\ell^m(x)$ ,  $m \geq 0$ , is defined by the following:

$$\mathbf{P}_\ell^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} (\mathbf{P}_\ell(x))$$

and we set  $\mathbf{P}_\ell^{-m}(x) := (-1)^m \frac{(\ell-m)!}{(\ell+m)!} \mathbf{P}_\ell^m(x)$ .

Let  $U_\epsilon$  denote the  $\epsilon$ -neighborhood of  $e$  in  $G$  for any  $\epsilon > 0$ .

**Lemma 5.2.** *Let  $\ell \geq 0$ . For all sufficiently small  $\epsilon > 0$ ,*

$$\phi_\ell(n_x a_y k) = (1 + O((\ell+1)^{n-1}\epsilon)) \cdot \phi_\ell(n_x a_y)$$

*with the implied constants independent of  $\ell$ ,  $x$ ,  $y > 0$  and  $k \in K_\epsilon := K \cap U_\epsilon$ .*

*Proof.* If  $n = 2$ ,  $\phi_\ell(n_x a_y k_\theta) = e^{2\ell i\theta} \phi_\ell(n_x a_y)$ , and hence the claim follows easily as  $e^{2\ell i\theta} = 1 + O((\ell+1)\theta)$  for  $\theta$  small.

Let  $n = 3$ . We set  $\mu = \mu_1 + \mu_2$ . One can compute explicitly that the Casimir operator  $\mathcal{C}_K$  acts on  $M$ -invariant functions as follow:

$$\mathcal{C}_K f(\mu, \theta) = \frac{1}{\sin^2 2\theta} \frac{\partial^2}{\partial \mu^2} f + \frac{1}{4 \sin 2\theta} \frac{\partial}{\partial \theta} \left( \sin 2\theta \frac{\partial}{\partial \theta} f \right).$$

Since  $\mathcal{C}_K(\phi_\ell) = -\ell(\ell+1)\phi_\ell$ , it follows from the theory of spherical harmonics (cf. [34]) that

$$\phi_\ell(n_x a_y k_{\mu_1, \mu_2 \theta}) = \sum_{m=-\ell}^{\ell} f_\ell^m(x, y) \cdot Y_\ell^m(\theta, \mu) \quad (5.3)$$

where

$$Y_\ell^m(\theta, \mu) = \frac{\sqrt{(2\ell+1)(\ell-m)!}}{\sqrt{4\pi(\ell+m)!}} \cdot \mathbf{P}_\ell^m(\cos(2\theta)) \cdot e^{im\mu},$$

and  $\mathbf{P}_\ell^m(x)$  is the associated Legendra function and  $f_{\ell, m} \in C^\infty(\mathbb{C} \times \mathbb{R}_{>0})$ .

We have for  $|\mu| < \epsilon$ ,  $e^{im\mu} = 1 + O(\ell\epsilon)$  as  $|m| \leq \ell$ . Also, from the properties of the Legendre functions, we deduce that for  $|\theta| < \epsilon$  and  $|m| \leq \ell$ ,

$$\mathbf{P}_\ell^m(\cos 2\theta) = (1 + O(\ell\epsilon))\mathbf{P}_\ell^m(1).$$

Therefore for  $|\theta| < \epsilon$  and  $|\mu| < \epsilon$ ,

$$\mathbf{P}_\ell^m(\cos 2\theta) \cdot e^{im\mu} = (1 + O(\ell^2\epsilon))\mathbf{P}_\ell^m(1),$$

and hence

$$Y_\ell^m(\theta, \mu) = (1 + O(\ell^2\epsilon))Y_\ell^m(0, 0).$$

It follows that for  $\ell \gg 1$ ,

$$\begin{aligned} \phi_\ell(n_x a_y k_{\mu_1, \mu_2 \theta}) &= (1 + O(\ell^2\epsilon)) \sum_{m=-\ell}^{\ell} f_\ell^m(x, y) \cdot Y_\ell^m(0, 0) \\ &= (1 + O(\ell^2\epsilon)) \phi_\ell(n_x a_y). \end{aligned}$$

□

Setting  $N^- := \left\{ \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \right\}$  where  $x$  ranges over  $\mathbb{R}$  (resp.  $\mathbb{C}$ ) for  $n = 2$  (resp.  $n = 3$ ), the product map

$$N \times A \times M \times N^- \rightarrow G$$

is a diffeomorphism at a neighborhood of  $e$ . Let  $dk$  be the invariant probability measure on  $K$  and denote by  $dg$  the Haar measure on  $G$ :  $dg = y^{-n} dx dy dk$  for  $g = n_x a_y k$ .

Let  $\nu$  be a smooth measure on  $AMN^-$  such that  $dn_x \otimes d\nu(a_y m n_{x'}^-) = dg$ . Fix a bounded open domain  $J \subset F_0$  which contains  $F_\Lambda$  and choose a compactly supported smooth function  $0 \leq \eta \leq 1$  on  $N$  with  $\eta|_J = 1$ . If  $\infty$  is of rank  $n - 1$ , then set  $J = F_0$ ; hence  $\eta = 1$  on  $F_0$ .

Fix  $\epsilon_0 > 0$  so that the multiplication map

$$\text{supp}(\eta) \times (U_{\epsilon_0} \cap AMN^-) \rightarrow \text{supp}(\eta) (U_{\epsilon_0} \cap AMN^-) \subset \Gamma \backslash G$$

is a bijection onto its image. For each  $0 < \epsilon < \epsilon_0$ , let  $0 \leq r_\epsilon \leq 1$  be a non-negative smooth function in  $AMN^-$  whose support is contained in  $W_\epsilon := (U_\epsilon \cap A)M(U_{\epsilon_0} \cap N^-)$  and  $\int_{W_\epsilon} r_\epsilon d\nu = 1$ .

We define the following function  $\rho_{\eta, \epsilon}$  on  $\Gamma \backslash G$ : for  $g = n_x a_y m n_{x'}^-$ ,

$$\rho_{\eta, \epsilon}(g) = \begin{cases} \eta(n_x) \cdot r_\epsilon(a_y m n_{x'}^-) & \text{for } g \in \text{supp}(\eta)W_\epsilon \\ 0 & \text{for } g \notin \text{supp}(\eta)W_\epsilon. \end{cases}$$

**Proposition 5.4.** *For any  $\ell \in \mathbb{Z}_{\geq 0}$ , we have for all  $0 < \epsilon \ll 1$ ,*

$$\phi_\ell^N(a_y) = \langle a_y \cdot \phi_\ell, \rho_{\eta, \epsilon} \rangle + O_\eta((\ell + 1)^{(3n-4)/2} \epsilon y^{n-1-\delta}) + O_\eta((\ell + 1)^{(n-2)/2} y^\delta)$$

*with the implied constants independent of  $\ell$ .*

*Proof.* Let  $h = a_{y_0} n_x^- m \in W_\epsilon$ . Then for  $n \in N$  and  $y > 0$ , we have

$$nha_y = na_{yy_0} n_{yx}^- m.$$

As the product map  $A \times N \times K \rightarrow G$  is a diffeomorphism and hence a bi-Lipschitz map in a neighborhood of  $e$ , there exists  $q \geq 1$  such that the  $\epsilon$ -neighborhood of  $e$  in  $G$  is contained in the product  $A_{q\epsilon} N_{q\epsilon} K_{q\epsilon}$  of  $q\epsilon$ -neighborhoods for all small  $\epsilon > 0$ .

Therefore we may write

$$n_{yx}^- = a_{y_1} n_{x_1} k_1 \in A_{qy\epsilon_0} N_{qy\epsilon_0} K_{\epsilon_0}$$

and hence

$$\begin{aligned} nha_y &= na_{yy_0 y_1} n_{x_1} k_1 m \\ &= n(a_{yy_0 y_1} n_{x_1} a_{yy_0 y_1}^{-1}) a_{yy_0 y_1} k_1 m = n(n_{x_1 y y_0 y_1}) a_{yy_0 y_1} k_1 m. \end{aligned}$$

By Lemma 5.2,

$$\phi_\ell(n_x ha_y) = \phi_\ell(n_x(n_{x_1 y y_0 y_1}) a_{yy_0 y_1})(1 + O((\ell + 1)^{n-1} \epsilon)).$$

Write  $\phi_\ell^N(a_y) = c_n(\ell) y^{n-1-\delta}$ . Hence, using Lemmas 5.1,

$$\begin{aligned} &\int_{(N \cap \Gamma) \setminus N} \phi_\ell(n_x ha_y) \cdot \eta(n_x) dx \\ &= (1 + O((\ell + 1)^{n-1} \epsilon)) \int_{N \cap \Gamma \setminus N} \phi_\ell(n_x(n_{x_1 y y_0 y_1}) a_{yy_0 y_1}) \cdot \eta(n_x) dx \\ &= (1 + O((\ell + 1)^{n-1} \epsilon)) \int_{N \cap \Gamma \setminus N} \phi_\ell(n_x a_{yy_0 y_1}) \cdot (\eta(n_x) + O(\epsilon)) dx \\ &= \int_{N \cap \Gamma \setminus N} \phi_\ell(n_x a_{yy_0 y_1}) \eta(n_x) dx + O_\eta(\epsilon(\ell + 1)^{n-1} \phi_\ell^N(a_{yy_0 y_1})) \\ &= c_n(\ell) (y y_0 y_1)^{n-1-\delta} + O_\eta(\epsilon(\ell + 1)^{n-1} (\ell + 1)^{\frac{n-2}{2}} y^{n-1-\delta}) + O_\eta((\ell + 1)^{\frac{n-2}{2}} y^\delta) \\ &= c_n(\ell) y^{n-1-\delta} + O((\ell + 1)^{\frac{3n-4}{2}} \epsilon y^{n-1-\delta}) + O_\eta((\ell + 1)^{\frac{n-2}{2}} y^\delta) \end{aligned}$$

as  $|y_0 - 1| = O(\epsilon)$  and  $|y_1 - 1| = O(y\epsilon)$ .

As  $\int r_\epsilon d\nu(h) = 1$ , we deduce from Lemma 5.1 that

$$\begin{aligned} \langle a_y \phi_\ell, \rho_{\eta, \epsilon} \rangle &= \int_{W_\epsilon} r_\epsilon(h) \left( \int_{N \cap \Gamma \setminus N} \phi_\ell(n_x ha_y) \eta(n_x) dx \right) d\nu(h) \\ &= c_n(\ell) y^{n-1-\delta} + O((\ell + 1)^{\frac{3n-4}{2}} \epsilon y^{n-1-\delta}) + O_\eta((\ell + 1)^{\frac{n-2}{2}} y^\delta). \end{aligned}$$

□

The above corollary implies:

**Corollary 5.5.** *For any  $\ell \geq 0$ ,*

$$|\langle a_y \cdot \phi_\ell, \rho_{\eta, \epsilon} \rangle| \ll (\ell + 1)^{\frac{3n-4}{2}} y^{n-1-\delta}$$

where the implied constant is independent of  $\ell \geq 0$ ,  $0 < \epsilon < 1$  and  $0 < y < 1$ .

## 6. EQUIDISTRIBUTION OF A CLOSED HOROSPHERE

As before, let  $n = 2$  or  $3$ , and  $\Gamma < G$  a geometrically finite discrete subgroup with  $(n-1)/2 < \delta < (n-1)$  and  $(N \cap \Gamma) \backslash N$  closed. In this section, we prove the main effective equidistribution theorem for  $(N \cap \Gamma) \backslash Na_y$  as  $y \rightarrow 0$ .

Let  $\{Z_i\}$  be a basis of the Lie algebra of  $G$ . For  $\psi \in C^\infty(\Gamma \backslash G) \cap L^2(\Gamma \backslash G)^M$ , we consider the following  $L^2$ -Sobolev norm  $\mathcal{S}_m(\psi)$ :

$$\mathcal{S}_m(\psi) = \max\{\|Z_{i_1} \cdots Z_{i_m}(\psi)\|_2\}.$$

By Theorem 3.1, we can fix  $(n-1)/2 < s_1 < \delta$  so that there is no eigenvalue of the Laplacian between  $s_1(n-1-s_1)$  and  $\delta(n-1-\delta)$  in  $L^2(\Gamma \backslash G)$ .

**Lemma 6.1.** *For any  $\psi_1, \psi_2 \in L^2(\Gamma \backslash G)^M \cap C^\infty(\Gamma \backslash G)$  and  $0 < y < 1$ , we have*

$$\langle a_y \cdot \psi_1, \psi_2 \rangle = \sum_{\ell \in \hat{K}} \langle \psi_1, \phi_\ell \rangle \langle a_y \cdot \phi_\ell, \psi_2 \rangle + O(y^{n-1-s_1} \cdot \mathcal{S}_{n-1}(\psi_1) \cdot \mathcal{S}_{n-1}(\psi_2)).$$

*Proof.* We have  $L^2(\Gamma \backslash G) = V \oplus V^\perp$  where  $V^\perp$  does not contain any complementary series with parameter  $s > \delta$ . Since  $V^M = \oplus_{\ell \in \hat{K}} \mathbb{C} \phi_\ell$ , we can write

$$\psi_1 = \sum_{\ell \in \hat{K}} \langle \psi_1, \phi_\ell \rangle \phi_\ell + \psi_1^\perp$$

with  $\psi_1^\perp \in V^\perp$  since  $\langle \psi_1 - \sum_{\ell \in \hat{K}} \langle \psi_1, \phi_\ell \rangle \phi_\ell, \phi \rangle = 0$  for any  $\phi \in V$ . Hence

$$\langle a_y \cdot \psi_1, \psi_2 \rangle = \sum_{\ell \in \hat{K}} \langle \psi_1, \phi_\ell \rangle \langle a_y \cdot \phi_\ell, \psi_2 \rangle + \langle a_y \cdot \psi_1^\perp, \psi_2 \rangle.$$

On the other hand, by the assumption on  $s_1$ , we have

$$|\langle a_y \cdot \psi_1^\perp, \psi_2 \rangle| \ll y^{n-1-s_1} \cdot \mathcal{S}_{n-1}(\psi_1) \cdot \mathcal{S}_{n-1}(\psi_2)$$

(see [20, Prop. 3.3] and [19, Cor. 5.6]).  $\square$

We refer to [19, Lem 6.5] for the next lemma:

**Lemma 6.2.** *For  $\psi \in C_c^\infty(\Gamma \backslash G)$ , there exists  $\hat{\psi} \in C_c^\infty(\Gamma \backslash G)$  such that*

(1) for all small  $\epsilon > 0$ , and  $h \in U_\epsilon$ ,

$$|\psi(g) - \psi(gh)| \leq \epsilon \cdot \widehat{\psi}(g)$$

for all  $g \in \Gamma \backslash G$ .

(2) for any  $i \geq 1$ ,  $\mathcal{S}_i(\widehat{\psi}) \ll \mathcal{S}_{2n-1}(\psi)$  where the implied constant depends only on  $\text{supp}(\psi)$ .

**Lemma 6.3.** For any  $\psi \in C_c^\infty(\Gamma \backslash G)$   $\ell \geq 0$ , and  $i \geq 1$ ,

$$|\langle \psi, \phi_\ell \rangle| \ll (\ell + 1)^{-2i} \|\mathcal{C}_K^i \psi\|_2.$$

In particular, for any fixed  $i \geq 1$ ,

$$\sum_{\ell \in \hat{K}} (\ell + 1)^{2i} |\langle \psi, \phi_\ell \rangle| \ll \|\mathcal{C}_K^{i+1} \psi\|_2.$$

*Proof.* Recall that the Casimir operator  $\mathcal{C}_K$  acts on each  $V_\ell$  as a scalar, say,  $\alpha_\ell$ . Then  $\alpha_\ell = -4\ell^2$  for  $n = 2$  and  $\alpha_\ell = -\ell(\ell + 1)$  for  $n = 3$ . Moreover, for any smooth vectors  $v, w$ ,  $\langle \mathcal{C}_K v, w \rangle = \langle v, \mathcal{C}_K w \rangle$ , that is,  $\mathcal{C}_K$  is an adjoint operator. Therefore

$$\langle \mathcal{C}_K^i \psi, \phi_\ell \rangle = \langle \psi, \mathcal{C}_K^i \phi_\ell \rangle = \alpha_\ell^i \langle \psi, \phi_\ell \rangle.$$

Hence for all  $i \geq 1$ ,

$$|\langle \psi, \phi_\ell \rangle| \leq |\alpha_\ell|^{-i} \cdot \|\mathcal{C}_K \psi\|_2.$$

□

**Theorem 6.4.** For any  $\psi \in C_c^\infty(\Gamma \backslash G)^M$ ,

$$\psi^N(a_y) = \sum_{\ell \in \hat{K}} c_\ell \langle \psi, \phi_\ell \rangle y^{n-1-\delta} + \mathcal{S}_{2n-1}(\psi) O(y^{n-1-\delta + \frac{2s_\Gamma}{(2n+1)}}).$$

*Proof.* Fix  $\psi \in C_c^\infty(\Gamma \backslash G)^M$ . When  $\infty$  has rank  $n - 1$ , we set  $J = F_0$ . In other cases, it was shown in [19] that there exists a bounded open subset  $J$  of  $F_0$  such that  $\psi(n_x a_y) = 0$  for all  $x \in F_0 - J$  and all  $0 < y < 1$ . We assume that  $J$  contains  $F_\Lambda$ .

Choose a non-negative function  $\eta \in C_c^\infty(N \cap \Gamma \backslash N)$  such that  $\eta|_J = 1$ . Then

$$I_\eta(\psi)(a_y) := \int_{(N \cap \Gamma) \backslash N} \psi(n_x a_y) \eta(n_x) dx = \psi^N(a_y).$$

Let  $\epsilon_0, W_\epsilon, r_\epsilon, \rho_{\eta, \epsilon}$  be as defined in section 5 with respect to this  $J$  and  $\eta$ . Since  $r_\epsilon$  is the approximation of the identity in  $A$  direction,  $\mathcal{S}_2(\rho_{\eta, \epsilon}) = O_\eta(\epsilon^{(-2n+1)/2})$ . For any  $0 < y < 1$ , and any small  $\epsilon > 0$ , we have (see the proof of Prop. 6.6 in [19])

$$|I_\eta(\psi)(a_y) - \langle a_y \cdot \psi, \rho_{\eta, \epsilon} \rangle| \ll (\epsilon + y) \cdot I_\eta(\widehat{\psi})(a_y). \quad (6.5)$$

Setting  $\psi_0(g) := \psi(g)$ , we define for  $1 \leq i \leq k$ , inductively

$$\psi_i(g) := \widehat{\psi}_{i-1}(g)$$

where  $\widehat{\psi}_{i-1}$  is given by Lemma 6.2.

Fix  $1 < s_1 < \delta$  so that there is no eigenvalue of the Laplacian between  $s_1(n-1-s_1)$  and  $\delta(n-1-\delta)$  in  $L^2(\Gamma \backslash G)$ , and let  $k$  be an integer bigger than  $1 + \frac{(n-1-\delta)(2n+1)}{2(\delta-s_1)}$ .

Applying Lemma 6.2 to each  $\psi_i$ , we obtain for  $0 \leq i \leq k-1$ ,

$$\begin{aligned} I_\eta(\psi_i)(a_y) &= \langle a_y \cdot \psi_i, \rho_{\eta, \epsilon} \rangle + O\left((\epsilon + y) \cdot I_\eta(\widehat{\psi}_i)(a_y)\right) \\ &= \langle a_y \cdot \psi_i, \rho_{\eta, \epsilon} \rangle + O\left((\epsilon + y) \cdot I_\eta(\psi_{i+1})(a_y)\right) \end{aligned}$$

and

$$I_\eta(\psi_k)(a_y) = \langle a_y \cdot \psi_k, \rho_{\eta, \epsilon} \rangle + O_\eta((\epsilon + y)\mathcal{S}_{n-1}(\psi_k)).$$

By Lemma 6.3,

$$|\langle \psi_i, \phi_\ell \rangle| = O(\ell + 1)^{-4} \mathcal{S}_4(\psi_i)$$

and by Lemma 6.2,  $\mathcal{S}_j(\psi_i) \ll \mathcal{S}_{2n-1}(\psi)$  for all  $j \geq 1$ .

Since  $|\langle a_y \cdot \phi_\ell, \rho_{\eta, \epsilon} \rangle| \ll \ell^{(3n-4)/2} y^{n-1-\delta}$  by Corollary 5.5, we have

$$\sum_{\ell \in \hat{K}} \langle \psi_i, \phi_\ell \rangle \langle a_y \cdot \phi_\ell, \rho_{\eta, \epsilon} \rangle = O(\mathcal{S}_{2n-1}(\psi_i) \cdot y^{n-1-\delta}).$$

By Lemma 6.1, we deduce that for each  $1 \leq i \leq k-1$ ,

$$\begin{aligned} \langle a_y \cdot \psi_i, \rho_{\eta, \epsilon} \rangle &= \sum_{\ell \in \hat{K}} \langle \psi_i, \phi_\ell \rangle \langle a_y \cdot \phi_\ell, \rho_{\eta, \epsilon} \rangle + O(y^{n-1-s_1} \cdot \mathcal{S}_{n-1}(\psi_i) \cdot \mathcal{S}_{n-1}(\rho_{\eta, \epsilon})) \\ &= O(\mathcal{S}_{2n-1}(\psi) \cdot y^{n-1-\delta}) + O(y^{n-1-s_1} \cdot \mathcal{S}_{n-1}(\psi_i) \mathcal{S}_{n-1}(\rho_{\eta, \epsilon})) \\ &= \mathcal{S}_{2n-1}(\psi) \cdot O(y^{n-1-\delta} + \epsilon^{-(2n-1)/2} y^{n-1-s_1}). \end{aligned}$$

Hence for any  $0 < y < \epsilon$ , using Proposition 5.4, we deduce

$$\begin{aligned} I_\eta(\psi)(a_y) &= \langle a_y \cdot \psi, \rho_{\eta, \epsilon} \rangle + \sum_{j=1}^{k-1} O\left(\langle a_y \cdot \psi_j, \rho_{\eta, \epsilon} \rangle (\epsilon + y)^j\right) + O_\psi((\epsilon + y)^k) \\ &= \langle a_y \cdot \psi, \rho_{\eta, \epsilon} \rangle + \mathcal{S}_{2n-1}(\psi) O(\epsilon \cdot y^{n-1-\delta} + \epsilon^{-(2n-1)/2} y^{n-1-s_1} + \epsilon^k) \\ &= \sum_{\ell \in \hat{K}} \langle \psi, \phi_\ell \rangle \langle a_y \cdot \phi_\ell, \rho_{\eta, \epsilon} \rangle + \mathcal{S}_{2n-1}(\psi) O(\epsilon \cdot y^{n-1-\delta} + \epsilon^{-(2n-1)/2} y^{n-1-s_1} + \epsilon^k) \\ &= \sum_{\ell \in \hat{K}} \langle \psi, \phi_\ell \rangle (c_\ell y^{n-1-\delta} + O_\eta((\ell + 1)^{(3n-4)/2} \epsilon y^{n-1-\delta}) + O_\eta((\ell + 1)^{(n-2)/2} y^\delta)) \\ &\quad + \mathcal{S}_{2n-1}(\psi) O(y^\delta + \epsilon y^{n-1-\delta} + \epsilon^{-(2n-1)/2} y^{n-1-s_1} + \epsilon^k). \end{aligned}$$

Since  $|\langle \psi, \phi_\ell \rangle| \ll (\ell + 1)^{-4} O(\mathcal{S}_4(\psi))$ , we have

$$\sum_{\ell \in \hat{K}} \langle \psi, \phi_\ell \rangle (\ell + 1)^{(3n-4)/2} = O(1), \quad \sum_{\ell \in \hat{K}} \langle \psi, \phi_\ell \rangle (\ell + 1)^{(n-2)/2} = O(1).$$

Hence we deduce

$$I_\eta(\psi)(a_y) = \sum_{\ell \in \hat{K}} \langle \psi, \phi_\ell \rangle c_\ell y^{n-1-\delta} + \mathcal{S}_{2n-1}(\psi) O(y^\delta + \epsilon y^{n-1-\delta} + \epsilon^{-(2n-1)/2} y^{n-1-s_1} + \epsilon^k).$$

By equating  $\epsilon \cdot y^{n-1-\delta}$  and  $\epsilon^{-(2n-1)/2} y^{n-1-s_1}$  we put  $\epsilon = y^{2(\delta-s_1)/(2n+1)}$  and obtain

$$I_\eta(\psi)(a_y) = \sum_{\ell \in \hat{K}} \langle \psi, \phi_\ell \rangle c_\ell y^{n-1-\delta} + \mathcal{S}_{2n-1}(\psi) O(y^{n-1-\delta + \frac{2(\delta-s_1)}{2n+1}}).$$

□

## 7. COMPARING MAIN TERMS FROM DIFFERENT APPROACHES

The main term  $\sum_{\ell \in \hat{K}} c_\ell \langle \psi, \phi_\ell \rangle$  in Theorem 6.4 is related to the space average of  $\psi$  with respect to the Burger-Roblin measure (which we will call the BR measure for short) by the result of [29] and [19] (also see [25]).

Recall the Patterson-Sullivan measure  $\nu_j = \nu_j^\Gamma$  on the boundary and  $\phi_0 = \phi_0^\Gamma$  given by

$$\phi_0(x + jy) = \int_{\mathbb{R}^{n-1}} \left( \frac{(|u|^2 + 1)y}{|x - u|^2 + y^2} \right)^\delta d\nu_j(u)$$

from section 1. Note that

$$\phi_0^\Gamma(e) = |\nu_j^\Gamma|.$$

As before, we normalize  $\nu_j$  so that  $\|\phi_0\|_2 = 1$ .

For  $\xi \in \partial(\mathbb{H}^n)$  and  $z_1, z_2 \in \mathbb{H}^n$ , recall the Busemann function:

$$\beta_\xi(z_1, z_2) = \lim_{s \rightarrow \infty} d(z_1, \xi_s) - d(z_2, \xi_s)$$

where  $\xi_s$  is a geodesic ray tending to  $\xi$  as  $s \rightarrow \infty$ .

Using the identification of  $T^1(\mathbb{H}^n)$  and  $G/M$ , we give the definition of the Bowen-Margulis-Sullivan measure  $m^{\text{BMS}}$  on  $\Gamma \backslash G/M$ . For  $u \in T^1(\mathbb{H}^n)$ , we denote by  $u^+$  and  $u^-$  the forward and the backward endpoints of the geodesic determined by  $u$ , respectively. The correspondence  $u \mapsto (u^+, u^-, \beta_{u^-}(j, \pi(u)))$  gives a homeomorphism between the space  $T^1(\mathbb{H}^n)$  with  $(\partial(\mathbb{H}^n) \times \partial(\mathbb{H}^n) - \{(\xi, \xi) : \xi \in \partial(\mathbb{H}^n)\}) \times \mathbb{R}$  where  $\pi : G \rightarrow G/K = \mathbb{H}^n$  is the canonical projection. Define the measure  $\tilde{m}^{\text{BMS}}$  on  $G/M$ :

$$d\tilde{m}^{\text{BMS}}(u) = e^{\delta \beta_{u^+}(j, \pi(u))} e^{\delta \beta_{u^-}(j, \pi(u))} d\nu_j(u^+) d\nu_j(u^-) dt.$$



This measure is left  $\Gamma$ -invariant and hence induces a measure  $m^{\text{BMS}}$  on  $\Gamma \backslash G/M$ .

Roblin obtained the following interesting identity in his thesis [30]:

**Theorem 7.1** (Roblin). *For  $\delta > (n-1)/2$ ,*

$$\|\phi_0\|_2^2 = |m^{\text{BMS}}| \cdot \int_{\mathbb{R}^{n-1}} \frac{dx}{(1+|x|^2)^\delta}.$$

As we have normalized  $\nu_j$  so that  $\|\phi_0\|_2 = 1$  and  $\phi_0(j) = |\nu_j|$ , we deduce

$$\frac{1}{|m^{\text{BMS}}|} = \int_{\mathbb{R}^{n-1}} \frac{dx}{(1+|x|^2)^\delta}.$$

To describe the equidistribution result of  $(N \cap \Gamma) \backslash Na_y$  from [29], we recall the measure  $\tilde{m}_N^{\text{BR}}$  defined in the introduction: for  $\psi \in C_c(G/M)$ ,

$$\tilde{m}_N^{\text{BR}}(\psi) = \int_{KAN} \psi(ka_y n_x) y^{\delta-1} dx dy d\nu_j(k(0)).$$

The BR measure  $m_N^{\text{BR}}$  (associated to the stable horospherical subgroup  $N$ ) is the measure on  $\Gamma \backslash G/M$  induced from  $\tilde{m}_N^{\text{BR}}$ .

We define the measure  $\mu_N^{\text{PS}}$  on  $N$  by

$$d\mu_N^{\text{PS}}(n_x) = e^{-\delta\beta_x(j, x+j)} d\nu_j(x) = (1+|x|^2)^\delta d\nu_j(x).$$

This induces a measure on  $(N \cap \Gamma) \backslash N$  for which we use the same notation. Since  $\mu_N^{\text{PS}}$  is supported in  $(N \cap \Gamma) \backslash (\Lambda(\Gamma) - \{\infty\})$ , which is compact, we have  $\mu_N^{\text{PS}}((N \cap \Gamma) \backslash N) < \infty$ .

The following is proved by Roblin [29] when  $(N \cap \Gamma) \backslash N$  is compact and in [25] in general.

**Theorem 7.2.** *Let  $\delta > 0$  and  $(N \cap \Gamma) \backslash N$  closed. For any  $\psi \in C_c(\Gamma \backslash G)^M$ ,*

$$\lim_{y \rightarrow 0} y^{\delta-n+1} \cdot \psi^N(a_y) = \frac{\mu_N^{\text{PS}}(N \cap \Gamma \backslash N)}{|m^{\text{BMS}}|} m_N^{\text{BR}}(\psi).$$

Comparing the main terms of Theorem 7.2 and Theorem 6.4, and using Theorem 7.1, we deduce the following interesting identity of the Burger-Roblin measure considered as a distribution on  $\Gamma \backslash G$ :

**Theorem 7.3.** *Let  $\delta > (n-1)/2$ . For any  $\psi \in C_c^\infty(\Gamma \backslash G)$ ,*

$$\kappa_\Gamma \cdot m_N^{\text{BR}}(\psi) = \sum_{\ell \in \hat{K}} c_\ell \langle \psi, \phi_\ell \rangle$$

where  $\kappa_\Gamma = \int_{\mathbb{R}^{n-1}} \frac{dx}{(1+|x|^2)^\delta} \cdot \int_{n_x \in (N \cap \Gamma) \backslash N} (1+|x|^2)^\delta d\nu_j(x)$ .

Now Theorem 1.6 is a direct consequence of Theorem 7.3 and Theorem 1.3.

## 8. APPLICATION TO COUNTING IN SECTORS

Let  $n = 2$  or  $3$ . Let  $Q$  be a real quadratic form of signature  $(n, 1)$  and  $v_0 \in \mathbb{R}^{n+1}$  be a non-zero vector such that  $Q(v_0) = 0$ . Let  $\Gamma_0 < \mathrm{SO}_Q(\mathbb{R})^\circ$  be a geometrically finite discrete subgroup with  $\delta > (n-1)/2$ . Suppose that  $v_0\Gamma_0$  is discrete.

Let  $\|\cdot\|$  be *any* norm in  $\mathbb{R}^{n+1}$  and set  $B_T := \{v \in \mathbb{R}^{n+1} : \|v\| < T\}$ . Let  $G = \mathrm{PSL}_2(\mathbb{R})$  if  $n = 2$  and  $\mathrm{PSL}_2(\mathbb{C})$  if  $n = 3$ . Let  $\iota : G \rightarrow \mathrm{SO}_Q(\mathbb{R})$  be a representation so that the stabilizer of  $v_0$  in  $G$  via  $\iota$  is  $NM$ . Let  $\Gamma := \iota^{-1}(\Gamma_0)$ .

**8.1. Counting I.** For  $g \in G$ , we write  $k(g)$  for the  $K$ -coordinate of  $g$  in the Iwasawa decomposition  $G = NAK$ . As before,  $M = C_K(A)$ . Fixing a function  $f$  on  $M \backslash K$ , define the counting function  $F_T$  on  $\Gamma \backslash G$  by

$$F_T(g) = \sum_{\gamma \in N \cap \Gamma \backslash \Gamma} \chi_{B_T}(v_0 \gamma g) f(k(\gamma g))$$

where  $\chi_{B_T}$  denotes the characteristic function of  $B_T$ .

For  $k \in M \backslash K$  and  $\psi \in C_c(G)$ , define  $\psi^k \in C_c(G)^M$  by

$$\psi^k(g) = \int_{m \in M} \psi(gmk) dm.$$

Similarly, for  $\Psi \in C_c(\Gamma \backslash G)$ , we set  $\Psi^k(g) = \int_{m \in M} \Psi(gmk) dm$ .

**Lemma 8.1.** *For  $\Psi \in C_c(\Gamma \backslash G)$  and for any bounded Borel function  $f$  on  $K$ , we have*

$$\langle F_T, \Psi \rangle = \int_{k \in M \backslash K} f(k) \int_{y > \|v_0 k\| T^{-1}} \left( \int_{(N \cap \Gamma) \backslash N} \Psi^k(n_x a_y) dx \right) y^{-n} dy dk.$$

*Proof.*

$$\begin{aligned} & \langle F_T, \Psi \rangle \\ &= \int_{\Gamma \backslash G} \sum_{\gamma \in N \cap \Gamma \backslash \Gamma} \chi_{B_T}(v_0 \gamma g) f(k(\gamma g)) \Psi(g) dg \\ &= \int_{(N \cap \Gamma) \backslash G} \chi_{B_T}(v_0 g) f(k(g)) \Psi(g) dg \\ &= \int_{a_y k \in AK} \chi_{B_T}(v_0 a_y k) f(k) \left( \int_{(N \cap \Gamma) \backslash N} \Psi(n_x a_y k) dx \right) y^{-n} dy dk \\ &= \int_{k \in M \backslash K} \int_{y > \|v_0 k\| T^{-1}} f(k) \left( \int_{(N \cap \Gamma) \backslash N} \left( \int_{m \in M} \Psi(n_x a_y mk) dm \right) dx \right) y^{-n} dy dk \end{aligned}$$

where  $dm$  is the probability Haar measure on  $M$  and  $dk$  is the probability Haar measure on  $K$  (also understood as the probability invariant measure on  $M \setminus K$ ).  $\square$

By Theorem 1.6, for any  $\Psi \in C^\infty(\Gamma \setminus G)^M$  and  $k \in M \setminus K$ , we have, as  $y \rightarrow 0$ ,

$$\int_{(N \cap \Gamma) \setminus N} \Psi^k(n_x a_y) dx = \kappa_\Gamma \cdot m_N^{\text{BR}}(\Psi^k) \cdot y^{n-1-\delta} + \mathcal{S}_{2n-1}(\Psi) O(y^{(n-1-\delta) + \frac{2s_\Gamma}{2n+1}}).$$

Therefore, we deduce from Lemma 8.1:

**Theorem 8.2.** *Let  $\Psi \in C^\infty(\Gamma \setminus G)$  and  $f$  a bounded Borel function. Then, as  $T \rightarrow \infty$ ,*

$$\langle F_T, \Psi \rangle = \frac{\kappa_\Gamma}{\delta} \cdot \left( \int_{k \in M \setminus K} \frac{m_N^{\text{BR}}(\Psi^k) \cdot f(k)}{\|v_0 k\|^\delta} dk \right) \cdot T^\delta + O(\mathcal{S}_{2n-1}(\Psi) T^{\delta - \frac{2s_\Gamma}{2n+1}}).$$

**8.2. Counting II.** Let  $\Omega \subset K$  be a left  $M$ -invariant Borel subset such that the Patterson-Sullivan measure of the boundary of  $\Omega^{-1}(0)$  is zero.

Consider the associated sector:

$$S_T(\Omega) := \{v \in v_0 A \Omega : \|v\| < T\}.$$

Let  $\{\Gamma_d < \Gamma_0 : d \in I\}$  be a family of subgroups of finite index which satisfies  $\text{Stab}_{\Gamma_0} v_0 = \text{Stab}_{\Gamma_d} v_0$  and which has a uniform spectral gap, say,  $s_0$ .

Set

$$\Xi_{v_0}(\Gamma_0, \Omega) := \frac{\kappa_{\iota^{-1}(\Gamma_0)}}{\delta} \int_{k^{-1} \in \Omega} \frac{d\nu_j^\Gamma(k(0))}{\|v_0 k^{-1}\|^\delta}. \quad (8.3)$$

We deduce the following from Theorem 8.2 in this section:

**Theorem 8.4.** *Suppose that the boundary of  $\Omega^{-1}(0)$  is a proper algebraic subset. Then for any  $\gamma' \in \Gamma_0$ ,*

$$\#(v_0 \Gamma_d \gamma' \cap S_T(\Omega)) = \frac{\Xi_{v_0}(\Gamma_0, \Omega)}{[\Gamma_0 : \Gamma_d]} T^\delta + O(T^{\delta - \frac{8s_0}{n(n+9)(2n+1)}}).$$

*Proof.* Let  $\Gamma := \iota^{-1}(\Gamma_0)$  and  $\gamma_0 := \iota^{-1}(\gamma')$ , and let  $U_\epsilon$  be an  $\epsilon$ -neighborhood of  $e$  in  $G$ . By abuse of notation, we use the notation  $\Gamma_d$  to denote  $\iota^{-1}(\Gamma_d)$ . Since  $\delta > (n-1)/2$ ,  $\Gamma$  is Zariski dense and hence any proper algebraic subset of  $\partial(\mathbb{H}^n)$  has  $\nu_j$ -measure zero [13]. Hence  $\nu_j(\partial(\Omega^{-1}(0))) = 0$ . Moreover, for all sufficiently small  $\epsilon > 0$ , there exists an  $\epsilon$ -neighborhood  $K_\epsilon$  of  $e$  in  $K$  such that for  $\Omega_{\epsilon+} = \Omega K_\epsilon$  and  $\Omega_{\epsilon-} = \cap_{k \in K_\epsilon} \Omega k$ ,

$$\nu_j(\Omega_{\epsilon+}^{-1}(0) - \Omega_{\epsilon-}^{-1}(0)) = O(\epsilon). \quad (8.5)$$

By the strong wave front lemma [14, Theorem 4.1], there exists  $0 < \ell_0 < 1$  such that for  $T \gg 1$ ,

$$S_T(\Omega)U_{\ell_0\epsilon} \subset S_{(1+\epsilon)T}(\Omega_{\epsilon+}) \quad \text{and} \quad S_{(1-\epsilon)T}(\Omega_{\epsilon-}) \subset \cap_{u \in U_{\ell_0\epsilon}} S_T(\Omega)u.$$

Let  $\psi_\epsilon \in C_c^\infty(G)$  be a non-negative function supported in  $U_{\ell_0\epsilon}$  with integral one, and set

$$\Psi_{\Gamma_d, \epsilon}(g) = \sum_{\gamma \in \Gamma_d} \psi_\epsilon(\gamma g).$$

Define the counting function  $F_T^\Omega$  on  $\Gamma_d \backslash G$  by

$$F_T^\Omega(g) = \sum_{\gamma \in N \cap \Gamma \backslash \Gamma_d} \chi_{S_T(\Omega)}(v_0 \gamma g).$$

Note that  $F_T^\Omega(\gamma_0) = \#v_0\Gamma\gamma_0 \cap S_T(\Omega)$  and that

$$F_{(1-\epsilon)T}^{\Omega_{\epsilon-}}(\gamma_0 g) \leq F_T^\Omega(\gamma_0) \leq F_{(1+\epsilon)T}^{\Omega_{\epsilon+}}(\gamma_0 g)$$

for all  $g \in U_{\ell_0\epsilon}$ . On the other hand,

$$\int_{\Gamma_d \backslash G} F_{(1\pm\epsilon)T}^{\Omega_{\epsilon\pm}}(\gamma_0 g) \Psi_{\Gamma_d, \epsilon}(g) dg = \int_{\Gamma_d \backslash G} F_{(1\pm\epsilon)T}^{\Omega_{\epsilon\pm}}(g) \Psi_{\Gamma_d, \epsilon}(\gamma_0^{-1} g) dg.$$

Therefore, if we set  $\Psi_{\Gamma_d, \epsilon}^{\gamma_0}(g) := \Psi_{\Gamma_d, \epsilon}(\gamma_0^{-1} g)$ , we have

$$\langle F_{(1-\epsilon)T}^{\Omega_{\epsilon-}}, \Psi_{\Gamma_d, \epsilon}^{\gamma_0} \rangle \leq F_T^\Omega(\gamma_0) \leq \langle F_{(1+\epsilon)T}^{\Omega_{\epsilon+}}, \Psi_{\Gamma_d, \epsilon}^{\gamma_0} \rangle$$

where the inner product has taken place in  $L^2(\Gamma_d \backslash G)$ . Since the Patterson-Sullivan measure  $\nu_j^\Gamma$  is normalized so that  $\phi_0^\Gamma(e) = |\nu_j^\Gamma|$  and  $\|\phi_0^\Gamma\|_2 = 1$ , we note that

$$\nu_j^{\Gamma_d} = \frac{1}{\sqrt{[\Gamma : \Gamma_d]}} \nu_j^\Gamma$$

for all positive integer  $d$ . Therefore it follows that  $\kappa_{\iota^{-1}(\Gamma_d)} = \frac{1}{\sqrt{[\Gamma : \Gamma_d]}} \kappa_{\iota^{-1}(\Gamma)}$ , and that for any  $\Gamma$ -invariant continuous function  $f$ ,

$$m_{\Gamma_d, N}^{\text{BR}}(f) = \frac{1}{\sqrt{[\Gamma : \Gamma_d]}} m_{\Gamma, N}^{\text{BR}}(f).$$

We use the following (see [25, Prop. 6.2], or [19, Sec. 7]):

$$\int_{k \in \Omega} \frac{m_{\Gamma, N}^{\text{BR}}(\Psi_{\Gamma, \epsilon}^k)}{\|v_0 k\|^\delta} dk = \int_{k \in \Omega^{-1}} \frac{d\nu_j^\Gamma(k(0))}{\|v_0 k^{-1}\|^\delta} \cdot (1 + O(\epsilon)). \quad (8.6)$$

Since  $\dim(G) = n(n+1)/2$ , we compute  $\mathcal{S}_{2n-1}(\Psi_\epsilon) = O(\epsilon^{-(n^2+9n-4)/4})$ . Hence putting these together and using Theorem 8.2, we have

$$\begin{aligned} & \langle F_{(1\pm\epsilon)T}^{\Omega_{\epsilon\pm}}, \Psi_{\Gamma_d, \epsilon}^{\gamma_0} \rangle \\ &= \frac{1}{\delta[\Gamma : \Gamma_d]} \kappa_\Gamma \cdot (1 \pm \epsilon)^\delta \cdot T^\delta \int_{k \in \Omega_{\epsilon\pm}^{-1}} \frac{d\nu_j(k(0))}{\|v_0 k^{-1}\|^\delta} + O(\epsilon T^\delta + \epsilon^{-(n^2+9n-4)/4} T^{\delta - \frac{2s_0}{2n+1}}) \\ &= \frac{\kappa_\Gamma \cdot T^\delta}{\delta[\Gamma : \Gamma_d]} \int_{k \in \Omega_{\epsilon\pm}^{-1}} \frac{d\nu_j(k(0))}{\|v_0 k^{-1}\|^\delta} + O(\epsilon T^\delta + \epsilon^{-(n^2+9n-4)/4} T^{\delta - \frac{2s_0}{2n+1}}) \\ &= \frac{\kappa_\Gamma \cdot T^\delta}{\delta[\Gamma : \Gamma_d]} \int_{k \in \Omega^{-1}} \frac{d\nu_j(k(0))}{\|v_0 k^{-1}\|^\delta} + O(\epsilon T^\delta + \epsilon^{-(n^2+9n-4)/4} T^{\delta - \frac{2s_0}{2n+1}})(1 + \epsilon). \end{aligned}$$

Hence we deduce

$$F_T^\Omega(e) = \frac{\kappa_\Gamma \cdot T^\delta}{\delta[\Gamma : \Gamma_d]} \int_{k \in \Omega^{-1}} \frac{d\nu_j(k(0))}{\|v_0 k^{-1}\|^\delta} + O(T^{\delta - \frac{8s_0}{n(n+9)(2n+1)}})$$

by equating  $\epsilon^{-(n^2+9n-4)/4} T^{-\frac{2s_0}{(2n+1)}} = \epsilon$ . This finishes the proof of Theorem 8.4.  $\square$

8.3. Let  $\mathcal{P}$  be an Apollonian packing as in Theorem 1.1. Let

$$Q(x_1, x_2, x_3, x_4) = 2(x_1^2 + x_2^2 + x_3^2 + x_4^2) - (x_1 + x_2 + x_3 + x_4)^2$$

be the Descartes quadratic form, which has signature  $(3, 1)$ . Let  $\mathcal{A}$  denote the Apollonian group, i.e., the subgroup of  $O_Q(\mathbb{Z})$  generated by

$$\begin{aligned} S_1 &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}, & S_2 &= \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}, \\ S_3 &= \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix}, & S_4 &= \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \end{aligned}$$

The Apollonian subgroup  $\mathcal{A}$  has its critical exponent equal to  $\alpha$ , the residual dimension of  $\mathcal{P}$  and is a geometrically finite group (cf. [19]).

In [19, Sec. 2], it was shown that there exists a vector  $v_0$  with  $Q(v_0) = 0$ , whose coordinates are given by the curvatures of four mutually tangent circles of  $\mathcal{P}$  such that

$$N_T(\mathcal{P}) = \{v \in v_0 \mathcal{A} : \|v\|_{\max} < T\} + 3$$

for all  $T \gg 1$ .

Therefore Theorem 8.4 implies:

**Corollary 8.7.** *We have*

$$N_T(\mathcal{P}) = c_{\mathcal{P}} \cdot T^\alpha + O(T^{\alpha-2s_{\mathcal{A}}/63}).$$

where  $c_{\mathcal{P}} > 0$  is a constant.

Moreover, if we set  $\mathcal{A}_0 < \mathrm{SO}(Q)^\circ$  to be a torsion free finite index subgroup of  $\mathcal{A}$  and write  $v_0\mathcal{A}$  as the disjoint union  $\cup_{i=1}^m v_i\mathcal{A}_0$ , then

$$c_{\mathcal{P}} = \sum_{i=1}^m \Xi_{v_i}(\mathcal{A}_0, K). \quad (8.8)$$

On the other hand, it can be deduced from the main results in [26] that

$$\lim_{T \rightarrow \infty} \frac{N_T(\mathcal{P})}{T^\alpha} = c_A \cdot \mathcal{H}_\alpha(\mathrm{Res}(\mathcal{P}))$$

where  $c_A > 0$  is a constant independent of  $\mathcal{P}$  (cf. [27] for details). Therefore Theorem 1.1 follows from Corollary 8.7.

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